

Statistics 521, Problem Set 7

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Reading:

- Shorack, PFS, Chapter 4, sections 4.1 - 4.4, pages 65 - 85;
- Shorack, PFS, Chapter 5, sections 5.1 - 5.3, pages 87 - 101.

Due: Wednesday, November 23, 2016.

1. Let μ be a sigma-finite measure and let ν be a finite measure on (Ω, \mathcal{A}) . Set $\phi = \mu - \nu$; i.e. define $\phi : \mathcal{A} \rightarrow (-\infty, \infty]$ by $\phi(A) = \mu(A) - \nu(A)$.
 - (a) Show that ϕ is a signed measure.
 - (b) Show that

$$\phi(A) = \int_A (f - g)d(\mu + \nu)$$

for some measurable functions f and g , $g \in \mathcal{L}_1(\mu + \nu)$. Thus ϕ can be written in the canonical form of the signed measure discussed in example 4.1.1 and Problem Set 6, problem 5.

(c) Apply the results of Problem Set 6, problem 9 (i.e. PFS Exercise 4.1.2, page 67) to ϕ : compute ϕ^+ , ϕ^- , $|\phi|$, and $|\phi|(\Omega)$, assuming for the latter that μ is also a finite measure.

2. For probability measures P and Q on (Ω, \mathcal{A}) , define

$$d_{TV}(P, Q) = \sup_{A \in \mathcal{A}} |P(A) - Q(A)|.$$

You showed in problem 3, problem set #6 that $d_{TV}(P, Q) = (1/2) \int |p - q|d\mu$ for any measure μ dominating both P and Q ; i.e. $P \ll \mu$, $Q \ll \mu$.

- (a) Show that $d_{TV}(P, Q)$ does not depend on the choice of μ .
 - (b) Use the results of problem 1, part (c) to show that $d_{TV}(P, Q) = (1/2)|P - Q|(\Omega)$.
3. Suppose that ϕ is a sigma-finite signed measure and $X \in L_1(|\phi|)$. The integral $\int Xd\phi$ is defined by

$$\int Xd\phi = \int Xd\phi^+ - \int Xd\phi^-.$$

Show that $|\int Xd\phi| \leq \int |X|d|\phi|$.

4. PfS, Exercise 4.2.3, page 73: Flip a coin. If heads results, let X be a Uniform(0, 1) random variable; if tails results, let X be a Poisson(λ) random variable. The resulting distribution of X on \mathbb{R} is labeled ϕ .
- (a) Let μ denote Lebesgue measure on \mathbb{R} . Find the Lebesgue decomposition of ϕ with respect to μ ; that is, write $\phi = \phi_{ac} + \phi_s$.
- (b) Let ν be counting measure on $\{0, 1, 2, \dots\}$. Find the Lebesgue decomposition of ϕ with respect to ν .
5. PfS, Exercise 4.4.3, page 84: Let F be \nearrow , right-continuous and bounded on \mathbb{R} with $F(-\infty) = 0$. Define μ_F via $\mu_F((a, b]) = F(b) - F(a)$ for all $a < b$. Show that $\mu_F \ll \lambda$ if and only if F is an absolutely continuous function on \mathbb{R} (in the sense of PfS, Definition 4.4.2, page 80).
6. **Bonus problem:**
- (a) Let the Hellinger distance $H(P, Q)$ be defined for probability measures P, Q on a measurable space (Ω, \mathcal{A}) by

$$H^2(P, Q) \equiv \frac{1}{2} \int \{\sqrt{p} - \sqrt{q}\}^2 d\mu$$

where $p = dP/d\mu$ and $q = dQ/d\mu$ and μ is any measure dominating both P and Q : $P \ll \mu$ and $Q \ll \mu$. Show that $H(P, Q)$ does not depend on the choice of μ .

(b) Show that

$$H^2(P, Q) \leq d_{TV}(P, Q) \leq cH(P, Q)$$

for some absolute constant c . Find c explicitly.