Statistics 521, Problem Set 6

Wellner; 11/2/2016

Reminders:

- No lecture, MW, November 7 and 9 (November 11, Holiday)
- Make-up lecture 2: Wednesday November 16, 12:30 - 1:20, Low 106.
- Midterm exam: Monday, November 14.

Reading:

- Shorack, PfS, Chapter 4, sections 4.1 - 4.4, pages 65 - 85;
- Shorack, PfS, Chapter 5, sections 5.1 - 5.3, pages 87 - 97.

Due: Wednesday, November 16, 2016.

1. (a) Give an example of a sequence of random variables X_n, X (all defined on a common probability space $(\Omega, \mathcal{A}, P)$) satisfying $X_n \rightarrow_{a.s.} X$, but $E(X_n) \nrightarrow E(X)$.
   (b) Give an example of a sequence of non-negative random variables X_n, X on a common probability space satisfying $E(X_n) \rightarrow E(X)$ but $X_n \nrightarrow_{a.s.} X$.
   (c) Give an example of a sequence of random variables X_n, X satisfying $X_n \rightarrow_d X$, but $X_n \nrightarrow_{p,a.s.,1} X$.

2. PfS, Exercise 3.5.7, page 61, modified as follows: Suppose that $f_0, f_1, \ldots$ are ≥ 0, defined on a sigma-finite measure space $(\Omega, \mathcal{A}, \mu)$. (a) Suppose that $\int_\Omega f_n d\mu = 1$ for $n = 0, 1, \ldots$, and $f_n \rightarrow_{a.e.} f_0$ with respect to μ. Show that

$$\sup_{A \in \mathcal{A}} \Big| \int_A f_n d\mu - \int_A f_0 d\mu \Big| \rightarrow 0 \qquad \text{as } \ n \rightarrow \infty \, .$$

   (b) Show that the conclusion of (a) holds if just $f_n \rightarrow_\mu f_0$ and $\int_\Omega f_n d\mu \rightarrow \int_\Omega f_0 d\mu$.

3. Suppose that P, Q are two probability measures on the same measurable space $(\Omega, \mathcal{A})$ which are both absolutely continuous with respect to the measure μ with densities (Radon-Nikodym derivatives) p and

q respectively. Thus $P(A) = \int_A p d\mu$ and $Q(A) = \int_A q d\mu$ for $A \in \mathcal{A}$. Show that

$$d_{TV}(P,Q) \equiv \sup_{A\in\mathcal{A}} |P(A) - Q(A)| = \frac{1}{2}\int |p-q| d\mu = \int (p-q)^+ d\mu \,.$$

4. Suppose that $X_n \sim \text{Binomial}(n, p_n)$ for $n = 1, 2, \ldots$ with $np_n \to \lambda > 0$, and let P_n be the induced distribution of X_n on $\mathbb{R}$. Let $X_0 \sim \text{Poisson}(\lambda)$ and let P_0 be the corresponding induced distribution on $\mathbb{R}$. Use Scheffé's theorem to show that $d_{TV}(P_n, P_0) \to 0$ as $n \to \infty$.

5. Let $X_{n1}, \ldots, X_{nn}$ be independent, $X_{nk} \sim \text{Bernoulli}(p_{nk})$, and let $Y_n \sim \text{Poisson}(\sum_{k=1}^n p_{nk})$. Let P_n be the distribution of $\sum_{k=1}^n X_{nk}$ and let Q_n be the distribution of Y_n. Show that

$$d_{TV}(P_n, Q_n) \equiv \sup_{A\in\mathcal{B}} |P(S_n \in A) - P(Y_n \in A)| \le \sum_{k=1}^n p_{nk}^2.$$

Note that when $p_{nk} = p_n \to 0$ for all k and $np_n \to \lambda$, then $\sum_{k=1}^n p_{nk}^2 = np_n^2 = (np_n)^2/n = O(n^{-1})$.

Hint: Construct S_n and Y_n on a common probability space as follows: let $T_{nk} \sim \text{Poisson}(p_{nk})$, $k = 1, \ldots, n$ be independent, and let $Z_{nk} \sim \text{Bernoulli}(1 - (1 - p_{nk})e^{p_{nk}})$, $k = 1, \ldots, n$ be independent and independent of the T_{nk}'s. Define $X_{nk} = 1_{[T_{nk}\ge 1]} + 1_{[T_{nk}=0]}1_{[Z_{nk}=1]}$. Set $S_n = \sum_{k=1}^n X_{nk}$, $Y_n = \sum_{k=1}^n T_{nk}$. Check that $X_{nk} \sim \text{Bernoulli}(p_{nk})$, $Y_n \sim \text{Poisson}(\sum_1^n p_{nk})$, and

$$\begin{aligned} P(T_{nk} = 0, X_{nk} = 1) &= e^{-p_{nk}} - (1 - p_{nk}) \\ P(T_{nk} \ge 1, X_{nk} = 0) &= 0, \quad P(T_{nk} \ge 2) = 1 - e^{-p_{nk}} - p_{nk}e^{-p_{nk}}. \end{aligned}$$

Show that

$$d_{TV}(P_n, Q_n) \le P(S_n \ne Y_n) \le \sum_{k=1}^n P(X_{nk} \ne T_{nk}) \le \sum_{k=1}^n p_{nk}^2.$$

6. Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. Uniform$(0,1)$ random variables. Let $Y_n \equiv X_{n:1} \equiv \min_{1\le i\le n} X_i$ be the first order statistic of the first n of the X_i's.

(a) Compute the survival function $1 - F_{Y_n}$ of Y_n and show that $Y_n \to_d Y \sim$ exponential(1).
(b) Compute the density function f_{Y_n} of Y_n and show that $f_{Y_n}(y) \to f_Y(y) = e^{-y}$ for $y \geq 0$.
(c) Use (b) and Scheffé's theorem to show that

$$d_{TV}(P_{Y_n}, P_Y) = \frac{1}{2} \int_0^\infty |f_{Y_n}(y) - f_Y(y)| dy \to 0.$$

How fast is the convergence in the last display?

7. Suppose that $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex. Show that $h(x,t) \equiv tg(x/t)$ is a convex function on $\mathbb{R}^d \times (0, \infty)$.
Hint: First show that $h(cx, st) = ch(x,t)$ for any $c > 0$, $(x,t) \in \mathbb{R}^d \times (0, \infty)$, and hence $t^{-1}h(x,t) = h(x/t, 1)$.

8. For $s \in \mathbb{R} \cup \{\pm\infty\}$, $u, v \in \mathbb{R}^+ \equiv [0, \infty)$, and $\lambda \in [0,1]$, the Hölder mean (or generalized mean) $M_s(u, v; \lambda)$ of order s is defined by

$$M_s(u,v;\lambda) = \begin{cases} (\lambda u^s + (1-\lambda)v^s)^{1/s}, & s \neq 0,\ u, v > 0, \\ 0, & s < 0,\ uv = 0, \\ u^\lambda v^{1-\lambda}, & s = 0, \\ u \wedge v, & s = -\infty, \\ u \vee v, & s = +\infty. \end{cases}$$

(a) Interpret $M_s(u, v; \lambda)$ in terms of some function of the expected value of some random variable X.
(b) Show that for any $r < s$ the inequality $M_r(u, v; \lambda) \leq M_s(u, v; \lambda)$ holds for all $u, v \in \mathbb{R}$, $\lambda \in [0, 1]$. Thus

$$M_r(u,v;\lambda) \leq M_0(u,v;\lambda) \leq M_s(u,v;\lambda)$$

(In class on 10/31 we proved a related statement with $r = -1$ and $s = 1$.)

9. PfS, Exercise 4.1.2, page 67: Identify ϕ^+, ϕ^-, $|\phi|$ and $|\phi|(\Omega)$ in the context of the prototypical situation of example 4.1.1, page 66. Be sure to specify Ω^+ and Ω^-.

10. **Optional bonus problem:** Let X and Y be non-negative random variables.
(a) Show that Hölder's inequality can be rewritten as

$$E(X^{1/r}Y^{1/s}) \leq (EX)^{1/r} \cdot (EY)^{1/s} \quad \text{where} \quad r^{-1} + s^{-1} = 1.$$

(b) Show that the function $g(x, y) = x^{1/r}y^{1/s}$ is a concave function of (x, y); i.e. show that $(\partial^2/\partial x^2)f \leq 0$, $(\partial^2/\partial y^2)f \leq 0$, and

$$\left\{\frac{\partial^2}{\partial x \partial y}f(x, y)\right\}^2 - \frac{\partial^2}{\partial x^2}f(x, y)\frac{\partial^2}{\partial y^2}f(x, y) \leq 0.$$

(c) Use the bivariate version of Jensen's inequality to prove the form of Hölder's inequality given in (a).

11. **Optional bonus problem:** Let $X_1, X_2, \ldots$ be i.i.d. Exponential(λ) random variables (with distribution function $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$), and let $Y_n \equiv X_{n:n} \equiv \max_{1 \leq i \leq n} X_i$.
(a) Find the distribution function F_{Y_n} of Y_n and find a sequence b_n so that $Z_n \equiv Y_n - b_n \to_d Z$; identify the distribution function F_Z of Z.
(b) Find the density function f_{Z_n} of Z_n and show that $f_{Z_n}(z) \to f_Z(z)$ for all $z \in \mathbb{R}$.
(c) Use (b) and Scheffé's theorem to show that

$$d_{TV}(P_{Z_n}, P_Z) = \frac{1}{2}\int_0^\infty |f_{Z_n}(z) - f_Z(z)|dz \to 0.$$

How fast is the convergence in the last display?