

Statistics 521, Problem Set 3

Wellner; 10/12/16

Reading:

Shorack, PfS, Chapter 2, pages 21-36;

Durrett, *Probability*, sections 1.2-1.6, pages 9-36.

Due: Wednesday, October 19, 2016.

1. PfS, Exercise 2.2.1, page 28:

Suppose that $(\Omega, \mathcal{A}) = (R_2, \mathcal{B}_2)$ where \mathcal{B}_2 denotes the σ -field generated by all open subsets of the plane. Recall that this σ -field contains all sets of the form $B \times R$ and $R \times B$ for all $B \in \mathcal{B}$ where $B_1 \times B_2 \equiv \{(r_1, r_2) : r_1 \in B_1, r_2 \in B_2\}$. Now define measurable transformations $X_1(r_1, r_2) = r_1$ and $X_2(r_1, r_2) = r_2$. Then define $Z_1 \equiv \sqrt{X_1^2 + X_2^2}$ and $Z_2 \equiv \text{sign}(X_1 - X_2)$ where $\text{sign}(r) = 1, 0, -1$ according as r is $> 0, = 0, < 0$. Give geometric descriptions of the σ -fields $\mathcal{F}(Z_1)$, $\mathcal{F}(Z_2)$, and $\mathcal{F}(Z_1, Z_2)$.

2. PfS, Exercise 2.2.2, page 28:

Suppose that \mathcal{C} is a $\bar{\pi}$ -system. Suppose that \mathcal{V} is a vector space of functions with:

- (i) $1_C \in \mathcal{V}$ for all $C \in \mathcal{C}$.
 - (ii) If $A_n \in \mathcal{V}$ satisfy $A_n \nearrow A$, then $1_A \in \mathcal{V}$.
- (a) Show that $1_A \in \mathcal{V}$ for every $A \in \sigma[\mathcal{C}]$.
 - (b) Show that every simple function

$$\sum_1^m x_i 1_{A_i} \text{ is in } \mathcal{V}$$

whenever $m \geq 1$, $x_i \in R$, and $\sum_1^m A_i = \Omega$ with $A_i \in \sigma[\mathcal{C}]$.

- (c) Show that \mathcal{V} contains all $\sigma[\mathcal{C}]$ -measurable functions.

3. PfS, Exercise 2.3.1, page 29:

Let X_1, X_2, \dots denote measurable functions from $(\Omega, \mathcal{A}, \mu)$ to $(\bar{R}, \bar{\mathcal{B}})$.

- (a) If $X_n \rightarrow_{a.e.} X$, then $X = \tilde{X}$ a.e. for some measurable \tilde{X} .
- (b) If $X_n \rightarrow_{a.e.} X$ and μ is complete, then X itself is measurable.

4. PfS, Exercise 2.3.2, page 31:
 - (a) Show that in general \rightarrow_{μ} does not imply $\rightarrow_{a.e.}$.
 - (b) Give an example with $\mu(\Omega) = \infty$ where $\rightarrow_{a.e.}$ does not imply \rightarrow_{μ} .
5. PfS, Exercise 2.3.3, page 32.
show that $X_n \rightarrow_{\mu} X$ if and only if $X_n - X_m \rightarrow_{\mu} 0$.
6. **Bonus problem:** Prove Slutsky's theorem (Theorem 4.1, PfS page 34): If $X_n \rightarrow_d X$ and random variables Y_n and Z_n satisfy $Y_n \rightarrow_p Y$ and $Z_n \rightarrow_p b$, then $Y_n X_n + Z_n \rightarrow_d aX + b$.