

Statistics 521, Problem Set 2

Wellner; 10/5/2016

Reading: Shorack, PfS, Chapters 1-2, pages 15-20 and 21 - 28.

Durrett, Probability, pages 14-23 and pages 407-410.

Due: Wednesday, October 12, 2016.

1. PfS, Exercise 1.1.3, page 9 (and read the proof of the $\pi - \lambda$ theorem, Proposition 1.1.5, pages 9-10).
 - (a) The minimal λ -system generated by the class \mathcal{D} is denoted by $\lambda[\mathcal{D}]$. Show that $\lambda[\mathcal{D}]$ is equal to the intersection of all λ -systems containing \mathcal{D} .
 - (b) A collection \mathcal{A} of subsets of Ω is a σ -field if and only if it is both a π -system and a λ -system.
 - (c) Let \mathcal{C} be a π -system and let \mathcal{D} be a λ -system. Then $\mathcal{C} \subset \mathcal{D}$ implies that $\sigma[\mathcal{C}] \subset \mathcal{D}$.
2. PfS, Exercise 1.2.1, page 15. Let $(\Omega, \mathcal{A}, \mu)$ denote a measure space. Show that

$$\begin{aligned}\widehat{\mathcal{A}}_\mu &\equiv \{A : A_1 \subset A \subset A_2, A_1, A_2 \in \mathcal{A}, \mu(A_2 \setminus A_1) = 0\} \\ &= \{A \cup N : A \in \mathcal{A} \text{ and } N \subset (\text{some } B) \in \mathcal{A} \text{ with } \mu(B) = 0\} \\ &= \{A \Delta N : A \in \mathcal{A}, N \subset (\text{some } B) \in \mathcal{A} \text{ with } \mu(B) = 0\}\end{aligned}$$

and is a σ -field. Show that $(\Omega, \widehat{\mathcal{A}}_\mu, \widehat{\mu})$ is complete.

3. PfS, Exercise 1.2.3, page 16.

Suppose that μ on a field \mathcal{C} is σ -finite on \mathcal{C} and is extended to $\mathcal{A} = \sigma[\mathcal{C}]$; call the extension μ .

 - (a) For each $A \in \mathcal{A}$ with $\mu(A) < \infty$ and each $\epsilon > 0$ there exists a set $C = C_\epsilon \in \mathcal{C}$ such that $\mu(A \Delta C) < \epsilon$.
 - (b) Let μ denote counting measure on the integers. Then $\mathcal{C} = \{C : C \text{ or } C^c \text{ is finite}\}$ is a field. Determine $\sigma[\mathcal{C}]$. Show that the conclusion of part (a) fails for the set of even integers.

4. PfS, Exercise 1.2.4, page 16. (Nonmeasurable sets). Let Ω consist of the 16 values $1, \dots, 16$. (Think of them arranged in four rows of four values.) Let

$$\begin{aligned} C_1 &= \{1, 2, 3, 4, 5, 6, 7, 8\}, & C_2 &= \{9, 10, 11, 12, 13, 14, 15, 16\}, \\ C_3 &= \{1, 2, 5, 6, 9, 10, 13, 14\}, & C_4 &= \{3, 4, 7, 8, 11, 12, 15, 16\}. \end{aligned}$$

Let \mathcal{C} denote the field generated by $\{C_1, C_2, C_3, C_4\}$, and let $\mathcal{A} = \sigma[\mathcal{C}]$.

(a) Show that $\mathcal{A} \equiv \sigma[\mathcal{C}] \neq 2^\Omega$. (Note that 2^Ω contains $2^{16} = 65,536$ sets.)

(b) Let $\mu(C_i) = 1/2$, $1 \leq i \leq 4$, with $\mu(C_1C_3) = 1/4$. Show $\hat{\mathcal{A}}_\mu = \mathcal{A}$ with $2^4 = 16$ sets.

(c) Let $\mu(C_i) = 1/2$ for $i = 2, 3, 4$, with $\mu(C_2C_4) = 0$. Show that $\hat{\mathcal{A}}_\mu$ has $2^{10} = 1024$ sets.

(d) Illustrate proposition 2.1 below in the context of this exercise.

5. PfS, Exercise A.1.5, page 428 (or PfS(2000), 9.1.5, page 182): It is shown on page 428 that for $W_m \equiv \inf\{t > 0 : \mathbb{N}(t) = m\}$ where \mathbb{N} is a standard Poisson process on \mathbb{R}^+ with intensity ν , we have

$$1 - F_{W_m}(t) = P(W_m > t) = \sum_{k=0}^{m-1} (\nu t)^k e^{-\nu t} / k!.$$

By differentiating both sides of this identity show that W_m has density

$$f_{W_m}(t) = \nu(\nu t)^{m-1} e^{-\nu t} / \Gamma(m) \quad \text{for } t \geq 0.$$

Thus W_m has a Gamma(m, ν) density.

6. **Optional bonus problem:** Let $\Omega = \mathbb{Z}$ = the integers, and let \mathcal{A} be the collection of subsets A of \mathbb{Z} so that A or A^c is finite. Let $\mu(A) = 0$ in the first case, and let $\mu(A) = 1$ in the second. Show that μ has no extension to $\sigma(\mathcal{A})$.