

Statistics 521, Problem Set 1

Wellner; 9/28/16

Reading: Shorack, PfS, Chapter 1, pages 1 - 17;
Durrett, Probability, pages 1-13; 401-406
Due: Wednesday, October 5, 2016.

- (a) Suppose that $\{\mathcal{A}_n\}$ is an increasing sequence of algebras, i.e. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ for all $n \geq 1$. Show that $\cup_{n=1}^{\infty} \mathcal{A}_n$ is an algebra.
(b) Suppose that the \mathcal{A}_n of (a) are σ -algebras. Show by constructing a counter-example that $\cup_{n=1}^{\infty} \mathcal{A}_n$ need not be a σ -algebra.
- Proposition 1.1(b), PfS, page 3: There exists a minimal field, σ -field, or monotone class generated by (or containing) any specified class \mathcal{C} of subsets of Ω .
- PfS, Exercise 1.1.1, PfS, page 4: Let \mathcal{C}_1 and \mathcal{C}_2 denote two collections of subsets of the set Ω . If $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$ and $\mathcal{C}_2 \subset \sigma[\mathcal{C}_1]$, then $\sigma[\mathcal{C}_1] = \sigma[\mathcal{C}_2]$.
- PfS, Exercise 1.1.2, PfS, page 8. We always have $\mu(\liminf A_n) \leq \liminf \mu(A_n)$, while $\limsup \mu(A_n) \leq \mu(\limsup A_n)$ if $\mu(\omega) < \infty$.
- PfS(2000), Exercise 9.1.4, page 182; or PfS(2012), Exercise A.1.4, page 428. Suppose that $X_n \sim \text{Binomial}(n, p_n)$ where $np_n \rightarrow \lambda > 0$. Show that

$$P(X_n = k) \rightarrow \frac{\lambda^k}{k!} \exp(-\lambda) = P(Y = k)$$

where $Y \sim \text{Poisson}(\lambda)$; this implies that $X_n \rightarrow_d Y$. Can this be strengthened?

- Let $I = P(Z \geq 2) = .02275\dots$ where $Z \sim N(0, 1)$. Thus $I = \int h(x)f(x)dx$ where $f(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ is the standard normal density and $h(x) = 1_{[x \geq 2]}$. The basic Monte Carlo estimator is $\hat{I}_1 = n^{-1} \sum_{i=1}^n h(X_i)$ where X_1, \dots, X_n are i.i.d. $N(0, 1)$. Note that most observations are “wasted” in that they are not near the right tail. Now we try again with “importance sampling”: let g be the $N(3, 1)$ density. When we sample from g , the corresponding estimator of I is

given by $\hat{I}_2 = n^{-1} \sum_{i=1}^n f(X_i)h(X_i)/g(X_i)$.

(a) Show that for $n = 100$

$$\text{Var}(\hat{I}_1) = p(1-p)/n = .02275(1 - .02275)/100 = 0.0222.../100$$

where $p = I = P(Z \geq 2) = .02275....$

(b) Show that for $n = 100$

$$\text{Var}(\hat{I}_2) = n^{-1}\text{Var}_g(f(X)h(X)/g(X)) = .001805.../100,$$

and hence the variance has been reduced by a factor of 12.3.

Hint: You may compute

$$\int_{-\infty}^{\infty} \{h^2(x)f^2(x)/g(x)\} dx$$

numerically in (b).

7. (Optional bonus problem). Suppose that X is a random variable with density f and $h : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $E|h(X)| = \int_{\mathbb{R}} |h(x)|f(x)dx < \infty$. Suppose that we want to estimate (or approximate) $I = Eh(X) = \int_{\mathbb{R}} h(x)f(x)dx$ based on sampling Y_1, \dots, Y_n from a density g on \mathbb{R} , and then forming $\hat{I}_n = n^{-1} \sum_{i=1}^n h(Y_i)f(Y_i)/g(Y_i)$. Show that the choice $g(y) = |h(y)|f(y) / \int |h(z)|f(z)dz$ minimizes the variance of \hat{I}_n .