Statistics 521, Final Exam

Wellner; Wednesday, 12/12/2012, 2:30-4:20 PM

1. (30 points). **Define** five of the following eight terms:
   - (a) *Absolute continuity* of a signed measure ϕ with respect to a measure μ, **and** *singularity* of ϕ with respect to μ.
   - (b) The *product* $\sigma-$*field* $\mathcal{A} \times \mathcal{A}'$ for two measurable spaces $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$.
   - (c) *Almost sure convergence* of a sequence of random variables $\{X_n\}$.
   - (d) *Independent random variables* $X_1, \ldots, X_n$ and *independent events* $A_1, \ldots, A_n$.
   - (e) The *tail* $\sigma-$*field* of a sequence of random variables $X_1, X_2, \ldots$.
   - (f) A $\overline{\pi}-$system $\mathcal{C}$.
   - (g) A $\lambda-$system $\mathcal{D}$.
   - (h) *Khintchine - equivalent* sequences of random variables.

2. (30 points). Give careful **statements** of three of the following six theorems or results:

   - (a) The first Borel-Cantelli lemma.
   - (b) The Kolmogorov zero-one law.
   - (c) Feller's weak law of large numbers.
   - (d) The strong law of large numbers.
   - (e) The $\pi - \lambda$ theorem.
   - (f) Fatou's lemma.

Do **either** 3 **or** 4:

3. (30 points). Let $X : \Omega \to \mathbb{R}$ be a random variable defined on the probability space $(\Omega, \mathcal{A}, P)$, let P_X denote the induced distribution of X on $(\mathbb{R}, \mathcal{B})$, and let g be a measurable function from $\mathbb{R}$ to $\mathbb{R}$.
(a) State the *theorem of the unconscious statistician* in this context.
(b) Sketch a proof of the theorem you stated in (a).

4. (30 points) Suppose that X and Y are independent random variables and that f and g are real-valued measurable functions from $(\mathbb{R}, \mathcal{B})$ to $(\mathbb{R}, \mathcal{B})$ such that $f(X)$ and $g(Y)$ are measurable. Suppose that $E|f(X)| < \infty$ and $E|g(Y)| < \infty$. Show that
$$E[f(X)g(Y)] = E[f(X)]E[g(Y)] \tag{1}$$

Do **either** 5 **or** 6:

5. (30 points). Suppose that $X, X_1, X_2, \ldots$ are independent and identically distributed random variables.
(a) Show that the following identities holds: for all $\lambda > 0$
$$P(\max_{1 \le k \le n} |X_k| > \lambda) = P(|X| > \lambda) \sum_{k=1}^{n} P(|X| \le \lambda)^{k-1} = 1 - P(|X| \le \lambda)^n.$$
[Hint: For the first identity use the same type of decomposition of the event on the left side as we used in the proof of Kolmogorov's inequality.]
(b) Use the identities in (a) to show that for $\epsilon > 0$
$$P(\max_{1 \le k \le n} |X_k| > n\epsilon) \begin{cases} \le nP(|X| > n\epsilon) \\ \ge 1 - \exp(-nP(|X| > n\epsilon). \end{cases}$$
(c) Use the results of (b) to show that $M_n \equiv n^{-1} \max_{1 \le k \le n} |X_k| \to_p 0$ if and only if $xP(|X| > x) \to 0$ as $x \to \infty$ (i.e. X is weak-L_1).

6. (30 points). Give an example of a distribution function F with density function f with respect to Lebesgue measure λ such that $E|X| = \infty$ but $\tau(x) \equiv xP(|X| > x) \to 0$ as $x \to \infty$. Thus if $X_1, \ldots, X_n$ are i.i.d. F, the WLLN holds: $\overline{X}_n - \mu_n \to_p 0$ for some sequence μ_n (where $\mu_n = E(X_1 1_{[|X_1| \le n]})$ works), but the strong law of large numbers fails: $\limsup_n |\overline{X}_n| = +\infty$ a.s.

Do **either** 7 **or** 8:

7. (30 points). Suppose that X_1 and X_2 are independent Rademacher random variables, and set $X_3 = X_1 X_2$. (Thus $P(X_j = \pm 1) = 1/2$ for $j = 1, 2$.)
(a) Show that X_3 is a Rademacher random variable: $P(X_3 = \pm) = 1/2$.
(b) Show that each pair of X_1, X_2, X_3 are independent random variables.
(c) Show that X_1, X_2, X_3 are *not* independent random variables.

8. (30 points). Let $(\Omega, \mathcal{A}, P)$ denote the probability space $([0,1], \mathcal{B} \cap [0,1], \lambda)$ where λ is Lebesgue measure. For $n = 1, 2, \ldots$ define

$$X_n(\omega) = \begin{cases} 1, & \text{if } 0 \le \omega < 1/3, \\ 2, & \text{if } 1/3 \le \omega < 1/3 + 2/3^n, \\ 3, & \text{if } 1/3 + 2/3^n \le \omega < 1. \end{cases}$$

(a) Are the X_n's independent?
(b) What is the tail $\sigma-$field of the X_n's?

Do **either** 9 **or** 10:

9. (30 points). Let $X_1, X_2, \ldots$ be i.i.d. with d.f. $F(x) = 1 - \exp(-x^\alpha)$ for $x \ge 0$ where $\alpha > 0$.
(a) Find a sequence b_n so that $\limsup_{n\to\infty}(X_n/b_n) = 1$ almost surely.
(b) Let $M_n \equiv \max_{1\le k\le n} X_k$. In the case $\alpha = 1$, find a sequence of numbers c_n so that $M_n - c_n \to_d$ "something" and find the distribution of "something".

10. (30 points). Suppose that $X_1, X_2, \ldots$ are uncorrelated and $E(X_j^2) \le M < \infty$ for all $j \ge 1$.
(a) Show that $\overline{X}_n - E(\overline{X}_n) \to_2 0$.
(b) Show that $\overline{X}_n - E(\overline{X}_n) \to_p 0$.
(c) Show that $n^\alpha(\overline{X}_n - E(\overline{X}_n)) \to_p 0$ for $0 < \alpha < \alpha_0$ for some α_0 (and determine α_0).