Statistics 521, Midterm Exam

Wellner; 11/9/2007

1. (24 points). **Define** *three* of the following five terms:
   (a) Convergence in probability of a sequence of random variables $\{X_n\}$.
   (b) Convergence in distribution of a sequence of random variables $\{X_n\}$.
   (c) $\liminf A_n$ for a sequence of events $\{A_n\}$.
   (d) A uniformly integrable sequence of random variables $\{X_n\}$
   (e) A *simple function* defined on a measurable space $(\Omega, \mathcal{A})$.

2. (20 points). Give careful **statements** of *two* of the following four theorems or results:
   (a) The monotone convergence theorem.
   (b) The Helly-Bray theorem.
   (c) Liapunov's inequality.
   (d) The Caratheodory extension theorem.

3. (30 points).
   (a) Suppose that X is a non-negative measurable function on a measurable space $(\Omega, \mathcal{A})$. Give an explicit sequence of simple functions X_n satisfying $X_n \nearrow X$.
   (b) Now suppose that $(\Omega, \mathcal{A}) = ((0,1), \mathcal{B}_{(0,1)})$, and that we give this measurable space the Lebesgue measure λ, which we call P since it is a probability measure on this $(\Omega, \mathcal{A})$. Suppose that $X(\omega) = \omega^{-1/2}$ for $\omega \in (0,1)$.
   (b-1) For the simple functions X_n as given in (a), evaluate
   $$\lim_{n\to\infty} \int X_n dP = \lim_{n\to\infty} E(X_n).$$
   (b-2) Find the (induced) distribution function $F = F_X$ of X on $\mathbb{R}$.

4. (24 points). Let $X, Y \geq 0$ a.s. with $P(XY \geq 1) = 1$ and $P(\Omega) = 1$. Let $\mu_X = E(X)$, $\mu_Y = E(Y)$.
   (a) Show that $\mu_X \cdot \mu_Y \geq 1$.
   (b) Show that
   $$(1 + \mu_X^2)^{1/2} \leq E\{(1 + X^2)^{1/2}\} \leq 1 + \mu_X.$$

5. (30 points). Consider the sequence of random variables $\{X_n\}_{n\geq 1}$ with df's
   $$F_n(x) = P(X_n \leq x) = \begin{cases} (1 - n^{-2})x, & 0 \leq x \leq 1, \\ (1 - n^{-2}), & 1 \leq x < n, \\ 1, & n \leq x < \infty. \end{cases}$$
   (a) Does $X_n \to_d$ "some" X? If so, what is the distribution function F of X?
   (b) Does $\exp(\sin(\pi X_n/2)) \to_d$ something? If so, what is something?
   (c) Compute EX_n^r for $r > 0$ and $n = 1, 2, \ldots$.
   (d) For what values of $r > 0$ does $EX_n^r \to$ something finite?
   (e) For what values of $r > 0$ is $\{X_n^r\}$ uniformly integrable?