

**Math/Stat 521, Final Exam**

Wellner; Wednesday, 12/14/2016, 2:30-4:20 PM

1. (28 points). **Define** four of the following six terms:
  - (a) The *product  $\sigma$ -field*  $\mathcal{A} \times \mathcal{A}'$  for two measurable spaces  $(\Omega, \mathcal{A})$  and  $(\Omega', \mathcal{A}')$ .
  - (b) *Almost sure convergence* of a sequence of random variables  $\{X_n\}$ .
  - (c) *Independent  $\sigma$ -fields and independent random variables*.
  - (d) The *tail  $\sigma$ -field* of a sequence of random variables  $X_1, X_2, \dots$ .
  - (e) *Khintchine - equivalent* sequences of random variables.
  - (f) A uniformly integrable sequence of random variables  $\{X_n\}$ .
  
2. (30 points). Give careful **statements** of *three* of the following six theorems or results:
  - (a) The Lebesgue decomposition theorem.
  - (b) The Kolmogorov zero-one law.
  - (c) The Fubini-Tonelli theorem.
  - (d) Kolmogorov's Strong Law of Large Numbers.
  - (e) Kolmogorov's maximal inequality.
  - (f) Vitali's theorem (three parts).

Do **either 3 or 4**:

3. (25 points).
  - (a) State two Borel-Cantelli lemmas.
  - (b) Prove the first Borel-Cantelli lemma.
  
4. (28 points). Suppose that  $X_1, X_2, \dots$  are independent and identically distributed random variables with  $E|X_1| < \infty$ .  
Let  $Y_n \equiv X_n 1\{|X_n| < n\}$  for  $n = 1, 2, \dots$ 
  - (a) Show that the sequences  $\{X_n\}$  and  $\{Y_n\}$  are Khintchine - equivalent.
  - (b) What does the result in (a) imply about  $P(X_n \neq Y_n \text{ i.o.})$ ?

Do **either 5 or 6**:

5. (28 points) Suppose that  $X_1, X_2, \dots$  are i.i.d. with  $E|X_1|^r < \infty$  for some  $r > 0$ . For  $n \geq 1$  let  $Y_n \equiv X_n 1_{\{|X_n| < n^{1/r}\}}$ .
- (a) Show that the sequences  $\{X_n\}$  and  $\{Y_n\}$  are Khinchine - equivalent.
- (b) What can you say about the sequence  $M_{n,r} \equiv n^{-1/r} \max_{1 \leq k \leq n} |X_k|$ ?
6. (24 points) Let  $X_1, \dots, X_n$  be i.i.d. and suppose that  $xP(|X_1| > x) \rightarrow 0$  as  $x \rightarrow \infty$ . For  $k \in \{1, \dots, n\}$ , let  $Y_{k,n} \equiv X_k 1_{\{|X_k| \leq n\}}$ .
- (a) Show that  $P(\cup_{k=1}^n [Y_{k,n} \neq X_k]) \rightarrow 0$  as  $n \rightarrow \infty$ .
- (b) State Feller's Weak Law of Large Numbers.

Do **either 7 or 8**:

7. (25 points). Suppose that  $P$  is the measure with density  $p(x) = (1/3)1_{[0,3]}(x)$  with respect to Lebesgue measure  $\lambda$  (on the Borel  $\sigma$ -field of  $\mathbb{R}$ ), and  $Q$  is the measure with density  $q(x) = (1/3)1_{[2,5]}(x)$  with respect to Lebesgue measure  $\lambda$ .
- (a) Is  $P \ll Q$ ? Why or why not?
- (b) Give the Lebesgue decomposition of  $P$  with respect to  $Q$ .
- (c) Is  $Q \ll P + Q$ ? Why or why not?
- (d) Give the Lebesgue decomposition of  $Q$  with respect to  $P + Q$ .
- (e) If  $\phi \equiv P - Q$ , find  $|\phi|(\mathbb{R})$ .
8. (25 points). Let  $P_\mu$  denote the distribution of a  $N(\mu, 1)$  random variable  $X$  on  $\mathbb{R}$ : thus  $(dP_\mu/d\lambda)(x) = \phi(x - \mu)$  where  $\lambda$  is Lebesgue measure on  $\mathbb{R}$  and  $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ .
- (a) Show that  $P_\mu \ll P_0$ .
- (b) Find the Radon-Nikodym derivative  $dP_\mu/dP_0$ .

Do **either 9 or 10**:

9. (30 points). (a) Suppose that  $H(x) = \int_{-\infty}^x h(t)dt$  where  $h(t) \geq 0$  for all  $t \in \mathbb{R}$ . Use the theorem of the unconscious statistician and Fubini's theorem to show that

$$EH(X) = \int_{-\infty}^{\infty} h(t)P(X \geq t)dt.$$

- (b) Suppose that  $X$  is a random variable with values in  $[1, \infty)$ ; i.e.  $P(X \geq 1) = 1$ . Let  $F$  be the distribution function of  $X$ :  $F(x) =$

$P(X \leq x)$ . Use the result of (a) (or another direct application of the theorem of the unconscious statistician and Fubini's theorem) to prove the following formula:

$$E \log X = \int_1^\infty (1 - F(t)) \frac{1}{t} dt.$$

(c) Give an example of a distribution function  $F$  of a random variable  $X$  such that  $E \log(X) < \infty$  but  $E(X^r) = \infty$  for all  $r > 0$ .

10. (30 points) Suppose that  $X$  and  $Y$  are independent random variables and that  $f$  and  $g$  are real-valued measurable functions from  $(\mathbb{R}, \mathcal{B})$  to  $(\mathbb{R}, \mathcal{B})$  such that  $f(X)$  and  $g(Y)$  are measurable. Suppose that we know that

$$E[f(X)g(Y)] = E[f(X)]E[g(Y)] \tag{1}$$

holds for  $f = 1_A$  and  $g = 1_B$  for sets  $A, B \in \mathcal{B}$ .

- (i) Show that (1) holds for  $f = 1_A$  with  $A \in \mathcal{B}$  and a non-negative (measurable) function  $g$ .
- (ii) Using the result of (i), show that (1) holds for  $f$  integrable (i.e.  $E|f(X)| < \infty$ ) and  $g \geq 0$  (and measurable).
- (iii) Using the result of (ii), show that (1) holds for  $f$  integrable and  $g$  integrable (and measurable).

Do **either** 11 **or** 12:

11. (27 points). Let  $X_1, X_2, \dots$  be i.i.d. with distribution function  $F$  given by

$$F(x) = \begin{cases} \frac{1}{2} \exp(-|x|^4/3), & x \leq 0 \\ 1 - \frac{1}{2} \exp(-|x|^4/3), & x \geq 0 \end{cases}, \tag{2}$$

and let  $M_n \equiv \max_{1 \leq k \leq n} X_k$ .

(a) Show that

$$\limsup_{n \rightarrow \infty} \frac{X_n}{(3 \log n)^{1/4}} = 1 \quad a.s.$$

(b) Show that  $M_n/(3 \log n)^{1/4} \rightarrow_{a.s.} 1$ .

12. (27 points) Let  $X_1, X_2, \dots$  be i.i.d. exponential (1) random variables; i.e.  $P(X_1 > x) = e^{-x}$  for all  $x \geq 0$ . Let  $M_n \equiv \max_{1 \leq k \leq n} X_k$ .
- (a) Show that  $\limsup_{n \rightarrow \infty} (M_n / \log n) = 1$  a.s.
  - (b) Show that  $M_n / \log n \rightarrow_{a.s.} 1$ .
  - (c) Show that  $M_n - \log n \rightarrow_d$  some  $Y$  and find the distribution function of  $Y$ .