Statistics 521, Midterm Exam

Wellner; 11/14/2016

1. (24 points). **Define** *three* of the following five terms:
   (a) $\limsup A_n$ for a sequence of events $\{A_n\}$, and give the common (intuitive) abbreviation for this set.
   (b) A measurable function $X : \Omega \to \Omega'$ where $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ are measurable spaces.
   (c) A *simple function* defined on a measurable space $(\Omega, \mathcal{A})$.
   (d) The Lebesgue integral $\int X d\mu$ of a (real-valued) measurable function X defined on a measure space $(\Omega, \mathcal{A}, \mu)$.
   (e) A Lebesgue - Stieltjes measure on the real line $\mathbb{R}$.

2. (24 points). Give careful **statements** of *three* of the following five theorems or results:
   (a) Fatou's lemma
   (b) A theorem relating Lebesgue-Stieltjes measures to (generalized) distribution functions.
   (c) A theorem relating convergence in measure ($\to_\mu$) to convergence almost everywhere ($\to_{a.e.}$).
   (d) Liapunov's inequality.
   (e) The Caratheodory extension theorem.

3. (36 points).
   (a) State Hölder's inequality for random variables X and Y.
   (b) State Jensen's inequality, and prove that $\left(\prod_{i=1}^n x_i\right)^{1/n} \le \{x_1 + \cdots + x_n\}/n$ for any real numbers $x_i \ge 0$ for $i \in \{1, \ldots, n\}$. (Be careful about the case with some $x_j = 0$.)
   (c) State Markov's inequality.
   (d) Use Markov's inequality or a special case to give a bound for $P(|\overline{X}_n - \mu| \ge t)$ when $X_1, \ldots, X_n$ are independent and identically distributed with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$ for all i. Does the resulting bound show that $\overline{X}_n \to_p \mu$ under these assumptions?

4. (30 points).
   (a) Give an example of a sequence of measurable functions (or random variables) $\{X_n\}$ defined on a probability space $(\Omega, \mathcal{A}, P)$ (which you should make explicit) for which $X_n \to_{a.s.} 0$ but $E(X_n) \to 0, 1,$ or $+\infty$ depending on the value of some number $c > 0$.
   (b) Give an example of a sequence of measurable functions (or random variables) $\{X_n\}$ defined on a probability space $(\Omega, \mathcal{A}, P)$ (which you should make explicit) for which $X_n \to_p 0$, but $X_n \not\to_{a.s.} 0$.
   (c) Give an example of a sequence of random variables $\{X_n\}$, X which satisfy $X_n \to_d X$ but for which $X_n \not\to_p X$ and $X_n \not\to_{a.s.} X$.

5. (42 points).
   (a) State the definition of $X_n \to_d X$ for random variables X_n and X.
   (b) State the Helly-Bray theorem.
   (c) What is the resulting equivalent formulation of $X_n \to_d X$ which results as a corollary of the Helly-Bray theorem.
   (d) State the Mann-Wald theorem.
   (e) Suppose that $X_n \sim$ Uniform on$\{1/n, 2/n, \cdots, n/n = 1\}$; i.e. $P(X_n = k/n) = 1/n$ for $k \in \{1, \ldots, n\}$. Show that $X_n \to_d X$ where $X \sim$ Uniform$(0,1)$.
   (f) Find a sequence of rv's $\{Y_n\}$ and Y, all on a common probability space, such that $Y_n \stackrel{d}{=} X_n$, $Y \stackrel{d}{=} X$, and $Y_n \to_{a.s.} Y$.

6. (30 points).
   (a) Suppose that X is a non-negative measurable function on a measurable space $(\Omega, \mathcal{A})$. Give an explicit sequence of simple functions X_n satisfying $X_n \nearrow X$.
   (b) Now suppose that $(\Omega, \mathcal{A}) = ((0,1), \mathcal{B}_{(0,1)})$, and that we give this measurable space the Lebesgue measure λ, which we call P since it is a probability measure on this $(\Omega, \mathcal{A})$. Suppose that $X(\omega) = -\log(\omega)$ for $\omega \in (0,1)$.

   For the simple functions X_n as given in (a), evaluate

$$\lim_{n\to\infty} \int X_n dP = \lim_{n\to\infty} E(X_n).$$

   (c) Find the (induced) distribution function $F = F_X$ of X on $\mathbb{R}$.