Statistics 521, Problem Set 9 Solutions

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1. PfS, Exercise 5.3.1, page 94.
   Prove that:
   (a) A function $\underline{X} = (X_1, X_2, \ldots) : \Omega \to R^\infty$ is $\mathcal{B}_\infty - \mathcal{A}-$measurable if and only if each X_n is $\mathcal{B} - \mathcal{A}-$measurable.
   (b) If $\underline{X} = (X_1, X_2, \ldots)$ is $\mathcal{B}^\infty - \mathcal{A}-$ measurable and if $(i_1, i_2, \ldots)$ is an arbitrary sequence of integers, then $\underline{Y} \equiv (X_{i_1}, X_{i_2}, \ldots)$ is $\mathcal{B}^\infty - \mathcal{A}-$measurable.

   Solution: (a) First suppose that $\underline{X} = (X_1, X_2, \ldots) : \Omega \to R^\infty$ is $\mathcal{B}_\infty - \mathcal{A}-$measurable. Fix $n \ge 1$. Then for any Borel set $B \in \mathcal{B}$ we have $X_n^{-1}(B) = \underline{X}^{-1}(R \times \cdots \times R \times B \times R \times \cdots) \in \mathcal{A}$, and hence X_n is $\mathcal{B} - \mathcal{A}-$measurable.
   Now suppose that X_n is $\mathcal{B} - \mathcal{A}-$measurable for each n. Since $\mathcal{B}^\infty = \sigma[\mathcal{C}_I]$, it suffices to show that $\underline{X}^{-1}(I) \in \mathcal{A}$ for any finite-dimensional rectangle $I = I_1 \times I_2 \times \ldots \times I_n \times R \times \ldots$ with each $I_j \in \mathcal{B}$. But we have

$$\begin{aligned}
\underline{X}^{-1}(I) &= \underline{X}^{-1}(I_1 \times I_2 \times \ldots \times I_n \times R \times \ldots) \\
&= \cap_{k=1}^n X_k^{-1}(I_k) \cap \cap_{k=n+1}^\infty X_k^{-1}(R) \\
&= \cap_{k=1}^n X_k^{-1}(I_k) \cap \Omega \\
&= \cap_{k=1}^n X_k^{-1}(I_k) \\
&\in \mathcal{A}
\end{aligned}$$

   since $X_k^{-1}(I_k) \in \mathcal{A}$ for each k by hypothesis.
   (b) From the "only if" part of (a), $\underline{X} = (X_1, X_2, \ldots)$ is $\mathcal{B}^\infty - \mathcal{A}-$ measurable implies that X_k is $\mathcal{B} - \mathcal{A}-$ measurable for each k, and hence X_{i_k} is $\mathcal{B} - \mathcal{A}-$ measurable for each k. Then by the "if" part of (a), $\underline{Y} \equiv (X_{i_1}, X_{i_2}, \ldots)$ is $\mathcal{B}^\infty - \mathcal{A}-$measurable.

2. PfS, Exercise 8.1.1, page 166.
   (a) Show that $P(AB) = P(A)P(B)$ if and only if $\{\emptyset, A, A^c, \Omega\}$ and $\{\emptyset, B, B^c, \Omega\}$ are independent $\sigma-$fields.

(b) Show that $A_1, \ldots, A_n$ are independent if and only if (for each $k = 1, \ldots, n$,

$$P(A_{i_1} \ldots A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j}) \qquad \text{whenever} \ \ 1 \le i_1 < \cdots < i_k \le n \,.$$

Solution: We will prove (b) first.
(b) Since $A_1, \ldots, A_n$ are independent if and only if the random variables $1_{A_1}, \ldots, 1_{A_n}$ are independent, if and only if the $\sigma-$fields $\mathcal{F}(1_{A_1}), \ldots, \mathcal{F}(1_{A_n})$ are independent, and of course these are just the $\sigma-$fields $\mathcal{A}_1 \equiv \{\emptyset, A_1, A_1^c, \Omega\}, \ldots, \mathcal{A}_n \equiv \{\emptyset, A_n, A_n^c, \Omega\}$.
It remains only to show that the $\sigma-$fields $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are independent if and only if for each $k = 1, \ldots, n$

$$P(A_{i_1} \cap \ldots \cap A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j}) \tag{1}$$

whenever $1 \le i_1 < \cdots < i_k \le n$. First suppose that the $\sigma-$fields $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are independent. Then by taking $\Omega \in \{\emptyset, A_j, A_j^c, \Omega\}$ for $j \in \{i_1, \ldots, i_k\}^c$, we have, with $B_j = A_{i_m}$ if $j = i_m$, $B_j = \Omega$ if $j \in \{i_1, \ldots, i_k\}^c$,

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(\cap_{j=1}^n B_j) = \prod_{j=1}^{n} P(B_j) = P(A_{i_1}) \cdots P(A_{i_k})$$

since $P(B_j) = P(\Omega) = 1$ for $j \in \{i_1, \ldots, i_k\}^c$; i.e. (1) holds. Now suppose that (1) holds. In particular, this implies that

$$P(B_1 \cap \ldots \cap B_n) = P(B_1) \cdots P(B_n) \tag{2}$$

where each $B_i \in \{\phi, A_i, \Omega\}$, $i = 1, \ldots, n$, since both sides are 0 if any $B_j = \emptyset$, and if $B_j = A_j$ for $j \in \{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$, $B_j = \Omega$ for $j \in \{j_1, \ldots, j_m\}^c \subset \{1, \ldots, n\}$, then (2) reduces to (1). Now consider replacing B_1 by the other remaining element of $\mathcal{A}_1$, A_1^c: if we replace B_1 by A_1^c, then since $\Omega = A_1 + A_1^c$ it follows that $\Omega \cap \cap_{k=2}^n A_k = \cap_{k=1}^n A_k + A_1^c \cap_{k=2}^n A_k$ and hence the left side of (2) becomes

$$\begin{aligned} P(\cap_{k=2}^n B_k) - P(A_1 \cap \cap_{k=2}^n B_k) &= P(B_2) \cdots P(B_n) - P(A_1) P(B_2) \cdots P(B_n) \\ &= (1 - P(A_1)) P(B_2) \cdots P(B_n) \\ &= P(A_1^c) P(B_2) \cdots P(B_n) \,; \end{aligned}$$

thus we have proved that

$$P(C_1 \cap B_2 \ldots \cap B_n) = P(C_1)P(B_2)\cdots P(B_n) \tag{3}$$

for $C_1 \in \mathcal{A}_1$, and $B_j \in \{\emptyset, A_j, \Omega\}$ for $j = 2, \ldots, n$. This is the first step of an induction argument. Now suppose that for some $k \in \{1, \ldots, n\}$.

$$P(C_1 \cdots C_{k-1} \cap B_k \cdots B_n) = P(C_1)\cdots P(C_{k-1})P(B_k)\cdots P(B_n) \tag{4}$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \ldots, k-1$, $B_i \in \{\emptyset, A_i, \Omega\}$, $i = k, \ldots, n$. Since $\Omega = A_k + A_k^c$ it follows that

$$\Omega \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^{n} B_j = A_k \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^{n} B_j + A_k^c \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^{n} B_j\,.$$

Thus upon replacing B_k by A_k^c on the left side of (4), we see that we have, since both $\Omega, A_k \in \{\emptyset, A_k, \Omega\}$,

$$\begin{aligned}
&P(\cap_{j=1}^{k-1}C_j \cap A_k^c \cap \cap_{j=k+1}^{n}B_j) \\
&= P(\cap_{j=1}^{k-1}C_j \cap \Omega \cap \cap_{j=k+1}^{n}B_j) - P(\cap_{j=1}^{k-1}C_j \cap A_k \cap \cap \cap_{j=k+1}^{n} B_j) \\
&= \prod_{j=1}^{k-1} P(C_j)P(\Omega) \prod_{j=k+1}^{n} P(B_j) - \prod_{j=1}^{k-1} P(C_j)P(A_k) \prod_{j=k+1}^{n} P(B_j) \\
&= \prod_{j=1}^{k-1} P(C_j)\cdot(1 - P(A_k))\cdot \prod_{j=k+1}^{n} P(B_j) \\
&= \prod_{j=1}^{k-1} P(C_j)\cdot P(A_k^c)\cdot \prod_{j=k+1}^{n} P(B_j)\,.
\end{aligned}$$

Hence we have proved that (4) implies that

$$P(C_1 \cdots C_k B_{k+1} \cdots B_n) = P(C_1)\cdots P(C_k)P(B_{k+1})\cdots P(B_n) \tag{5}$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \ldots, k$, $B_i \in \{\emptyset, A_i, \Omega\}$, $i = k+1, \ldots, n$. It then follows by induction that

$$P(C_1 \cdots C_n) = P(C_1)\cdots P(C_n) \tag{6}$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \ldots, n$; i.e. the $\sigma-$fields $\mathcal{A}_i$, $i = 1, \ldots, n$ are independent.

(a) This follows immediately from (a) with $n = 2$.

3. PfS, Exercise 8.1.3, page 168.
If $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are independent classes of events and if each $\mathcal{C}_i$ is a $\overline{\pi}$-system, then $\sigma[\mathcal{C}_1], \ldots, \sigma[\mathcal{C}_n]$ are independent $\sigma-$fields.

Solution: First Proof: Let $\mathcal{C}$ be the class of "product" sets

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n = \{C_1 \times \cdots \times C_n : \ C_i \in \mathcal{C}_i, \ i = 1, \ldots, n\} .$$

Note that $\mathcal{C}$ is a $\overline{\pi}-$system for the product space $\Omega \times \cdots \times \Omega$: if $A = A_1 \times \cdots \times A_n \in \mathcal{C}$ and $B = B_1 \times \cdots \times B_n \in \mathcal{C}$, then $A \cap B = A_1 \cap B_1 \times \cdots \times A_n \cap B_n \in \mathcal{C}$ since each $\mathcal{C}_i$ is a $\overline{\pi}-$system. Now define two measures on $(\Omega^n, \mathcal{A}^n) = (\Omega \times \cdots \times \Omega, \mathcal{A} \times \cdots \times \mathcal{A})$:

$$Q_1(A_1, \ldots, A_n) = P(\cap_{i=1}^n A_i); \qquad Q_2(A_1, \ldots, A_n) = \prod_{i=1}^n P(A_i) .$$

Then $Q_1 = Q_2$ on $\mathcal{C}$ by hypothesis. Hence by Dynkin's lemma, $Q_1 = Q_2$ on $\sigma[\mathcal{C}] = \sigma[\mathcal{C}_1] \times \cdots \sigma[\mathcal{C}_n]$, and hence $\sigma[\mathcal{C}_1], \ldots, \sigma[\mathcal{C}_n]$ are independent $\sigma-$fields.

Second Proof: Since $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are independent, if $C_i \in \mathcal{C}_i$ for $i = 1, \ldots, n$, then

$$P(C_1 \cap \ldots \cap C_n) = \prod_{j=1}^n P(C_j) .$$

Fix $C_2 \in \mathcal{C}_2, \ldots, C_n \in \mathcal{C}_n$, and define the collection $\mathcal{D}$ by

$$\mathcal{D} = \{D \in \mathcal{A} : P(D \cap C_2 \cap \ldots \cap C_n) = P(D) \prod_{j=2}^n P(C_j)\} .$$

Then we clearly have $\mathcal{C}_1 \subset \mathcal{D}$. Moreover, $\mathcal{D}$ is a $\lambda-$system since:
(a) $\Omega \in \mathcal{C}_1 \subset \mathcal{D}$.
(b) If $A, B \in \mathcal{D}$ with $B \subset A$, then

$$P(A \cap C_2 \cap \ldots \cap C_n) = P(A) \prod_{j=2}^n P(C_j) , \tag{7}$$

$$P(B \cap C_2 \cap \ldots \cap C_n) = P(B) \prod_{j=2}^n P(C_j) , \tag{8}$$

and $A = (A \setminus B) + B$, so we have

$$\begin{aligned}
P(A \cap C_2 \cap \ldots \cap C_n) &= P(A) \prod_{j=2}^{n} P(C_j) \\
&= P((A \setminus B + B) \cap C_2 \cap \ldots \cap C_n) \\
&= P((A \setminus B) \cap C_2 \cap \ldots \cap C_n) + P((B \cap C_2 \cap \ldots \cap C_n) \\
&= P((A \setminus B) \cap C_2 \cap \ldots \cap C_n) + P(B) \prod_{j=2}^{n} P(C_j)\,.
\end{aligned}$$

Hence, using (7) on the left side,

$$\begin{aligned}
P((A \setminus B) \cap C_2 \cap \ldots \cap C_n) &= P(A) \prod_{j=2}^{n} P(C_j) - P(B) \prod_{j=2}^{n} P(C_j) \\
&= (P(A) - P(B)) \prod_{j=2}^{n} P(C_j) \\
&= P(A \setminus B) \prod_{j=2}^{n} P(C_j)\,.
\end{aligned}$$

Thus $A \setminus B \in \mathcal{D}$.
(c) Suppose that $A_1, A_2, \ldots \in \mathcal{D}$, and $A_n \nearrow A$. Then, since A_k is $\nearrow$

implies that the collection $\{A_k \setminus A_{k-1}\}$ is disjoint,

$$\begin{aligned}
&P(A \cap C_2 \cap \ldots \cap C_n) \\
&\quad = P(\cup_{k=1}^{\infty} A_k \cap C_2 \cap \ldots \cap C_n) \\
&\quad = P(\cup_{k=1}^{\infty} (A_k \setminus A_{k-1}) \cap C_2 \cap \ldots \cap C_n) \\
&\quad = \sum_{k=1}^{\infty} P((A_k \setminus A_{k-1}) \cap C_2 \cap \ldots \cap C_n) \\
&\quad = \sum_{k=1}^{\infty} P(A_k \setminus A_{k-1}) P(C_2) \cdots P(C_n) \\
&\qquad \text{since } \; A_k \supset A_{k-1}, A_k, A_{k-1} \in \mathcal{D} \; \text{ implies } \; A_k \setminus A_{k-1} \in \mathcal{D} \text{ by } (b) \\
&\quad = \left(\sum_{k=1}^{\infty} P(A_k \setminus A_{k-1}) \right) P(C_2) \cdots P(C_n) \\
&\quad = P\left(\sum_{k=1}^{\infty} (A_k \setminus A_{k-1}) \right) P(C_2) \cdots P(C_n) \\
&\quad = P(A) P(C_2) \cdots P(C_n) \,.
\end{aligned}$$

Thus $A \in \mathcal{D}$, and we conclude that $\mathcal{D}$ is a $\lambda-$system. Thus by the $\pi - \lambda$ theorem we conclude that $\sigma[\mathcal{C}_1] \subset \mathcal{D}$. Since the sets $C_i \in \mathcal{C}_i$ for $i = 2, \ldots, n$ were arbitrary, this means that $\sigma[\mathcal{C}_1], \mathcal{C}_2, \ldots, \mathcal{C}_n$ are independent. Next, fix $C_1 \in \sigma[\mathcal{C}_1]$, $C_3 \in \mathcal{C}_3, \ldots, C_n \in \mathcal{C}_n$ and repeat the above argument to show that

$$\sigma[\mathcal{C}_1], \sigma[\mathcal{C}_2], \mathcal{C}_3, \ldots, \mathcal{C}_n$$

are independent. Repeating it $n-2$ more times yields the conclusion.

4. Give an example of two collections of sets $\mathcal{A}_1$ and $\mathcal{A}_2$ that are independent but the generated $\sigma-$fields are not independent.

   Solution: One example of this goes as follows: let $\Omega = \{1, 2, 3, 4\}$, and let $\mathcal{A} = 2^{\Omega}$. Suppose that $P(\{\omega\}) = 1/4$ for each $\omega \in \Omega$. Suppose that $\mathcal{A}_1 = \{\{1, 2\}\}$ and $\mathcal{A}_2 = \{\{2, 3\}, \{2, 4\}\}$. For $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$, we have

$$P(A_1 \cap A_2) = P(\{2\}) = 1/4 = (1/2)(1/2) = P(A_1)P(A_2)$$

so that $\mathcal{A}_1$, $\mathcal{A}_2$ are independent classes. Then we have $\{2\} \in \sigma[\mathcal{A}_2]$ (since $\{2\} = \{2,3\} \cap \{2,4\}$) and $\{1,2\} \in \sigma[\mathcal{A}_1]$, but

$$P(\{1,2\} \cap \{2\}) = P(\{2\}) = 1/4 \neq 1/8 = P(\{1,2\})P(\{2\})\,.$$

Thus $\sigma[\mathcal{A}_1]$ and $\sigma[\mathcal{A}_2]$ are *not* independent classes. The difficulty here is that the class $\mathcal{A}_2$ is not a $\overline{\pi}-$system.