**Statistics 521, Problem Set 4 Solutions**

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1. PfS, Exercise 2.3.4, page 32: (a) Suppose that $\mu(\Omega) < \infty$ and $g$ is continuous a.e. $\mu_X$. Then $X_n \to_\mu X$ implies $g(X_n) \to_\mu g(X)$.
(b) Let $g$ be uniformly continuous on the real line. Then $X_n \to_\mu X$ implies that $g(X_n) \to_\mu g(X)$. (Here $\mu(\Omega) = \infty$ is allowed.)

   **Solution:** (a) Let $\{n'\}$ be a subsequence. We want to show that for some subsequence $\{n''\}$ it follows that $g(X_{n''}) \to_{a.e.} g(X)$. Then by (15) of Theorem 2.3.1 it follows that $g(X_n) \to_\mu g(X)$. But since $X_n \to_\mu X$ we know, by (15) of Theorem 2.3.1, that there is a further subsequence $\{n''\}$ such that $X_{n''} \to_{a.e.} X$. For this subsequence we have $g(X_{n''}) \to_{a.e.} g(X)$ (by restricting in addition to the set $[X \in C_g]$ with $\mu([X \in C_g^c]) = \mu_X(C_g^c) = 0$ for which $g$ is continuous). Thus we conclude that $g(X_n) \to_\mu g(X)$.
(b) Let $\epsilon > 0$. Since $g$ is uniformly continuous there is a $\delta = \delta_\epsilon$ such that $|y - x| < \delta_\epsilon$ implies $|g(y) - g(x)| < \epsilon$. Since $X_n \to_\mu X$, for every $\gamma > 0$ there exists an $N = N_{\epsilon,\gamma}$ such that

$$\mu([|X_n - X| \ge \delta_\epsilon]) < \gamma, \qquad \text{for all } n \ge N_{\epsilon,\gamma}.$$

   Then we have

$$\begin{aligned} &\mu([|g(X_n) - g(X)| \ge \epsilon]) \\ &\quad = \mu([|g(X_n) - g(X)| \ge \epsilon] \cap [|X_n - X| \ge \delta_\epsilon]) \\ &\qquad\qquad + \mu([|g(X_n) - g(X)| \ge \epsilon] \cap [|X_n - X| < \delta_\epsilon]) \\ &\quad \le \mu([|X_n - X| \ge \delta_\epsilon]) + \mu(\emptyset) \\ &\quad \le \gamma + 0 = \gamma \qquad \text{for } \ n \ge N_{\epsilon,\gamma}. \end{aligned}$$

   Thus $\mu([|g(X_n) - g(X)| \ge \epsilon]) \to 0$ as $n \to \infty$; i.e. $g(X_n) \to_\mu g(X)$.

2. PfS, Exercise 3.2.1, page 42: If $X \ge 0$ and $\int X d\mu = 0$, then $\mu([X > 0]) = 0$.

   **Solution:** Let $\epsilon > 0$. Then $X \ge \epsilon 1_{[X \ge \epsilon]}$, and hence

$$0 = \int X d\mu \ge \epsilon \mu([X \ge \epsilon])$$

Since $[X > 0] = \cup_{n=1}^{\infty}[X \geq 1/n] = \lim[X \geq 1/n]$, We find that $\mu([X > 0]) = \lim_n \mu([X \geq 1/n]) = \lim_n 0 = 0$.

3. PfS, Exercise 3.2.2, page 42: Show that

$$\int_A X d\mu \begin{cases} = 0 \\ \geq 0 \end{cases} \quad \text{for all } A \in \mathcal{A}, \quad \text{implies} \quad X \begin{cases} = 0 \ a.e. \\ \geq 0 \ a.e. \end{cases}$$

**Solution:** Suppose first that $\int_A X d\mu = 0$ for all $A \in \mathcal{A}$. Then with $A = [X^+ \geq 0]$ we have $0 = \int X 1_{[X^+ \geq 0]} d\mu = \int Y d\mu$ where $Y \equiv X 1_{[X^+ \geq 0]} = X^+ \geq 0$. Then by the previous exercise, $0 = \mu([Y > 0]) = \mu([X^+ > 0])$; i.e $X^+ = 0$ a.e. Similarly, choosing $A = [X^- \geq 0]$ yields $X^- = 0$ a.e.; combining the two results gives $X = X^+ - X^- = 0$ a.e.
Now suppose that $\int_A X d\mu \geq 0$ for all $A \in \mathcal{A}$. Taking $A = [X < 0] = [X^- > 0]$ yields $0 \leq \int_A X d\mu = \int -X^- d\mu \leq 0$ since $X^- \geq 0$. Thus $\int X^- d\mu = 0$. By problem 2 this implies $\mu([X < 0]) = \mu([X^- > 0]) = 0$. Hence $X \geq 0$ a.e. $\mu$.

4. PfS, Exercise 3.2.4, page 43. Let $Y \equiv g(X)$ in the context of Theorem 3.2.6 (the "Theorem of the unconscious statistician"). Show that the second equality holds in:

$$\int_{X^{-1}(g^{-1}(B))} g(X(\omega)) d\mu(\omega) = \int_{g^{-1}(B)} g(x) d\mu_X(x) = \int_B y d\mu_Y(y) \quad \text{for } B \in \overline{\mathcal{B}}$$

where $\mu_Y$ is the induced measure of $Y$ on $(\overline{R}, \overline{\mathcal{B}})$.

**Solution:** This boils down to what is proved in the first equality with an appropriate identification of terms. Thus $(\Omega, \mathcal{A})$ is replaced by $(\Omega', \mathcal{A}')$, $(\Omega', \mathcal{A}')$ is replaced by $(\overline{R}, \overline{\mathcal{B}})$, $g$ is replaced by the identity function $h : (\Omega', \mathcal{A}') \mapsto (\overline{R}, \overline{\mathcal{B}})$ given by $h(v) = v$, and $X$ is replaced by $g : (\Omega', \mathcal{A}') \mapsto (\overline{R}, \overline{\mathcal{B}})$, and $\mu$ and $\mu_X$ are replaced by $\mu_X$ and $\mu_Y$ respectively. Thus we think of the identity

$$\int g(X(\omega)) d\mu(\omega) = \int g(x) d\mu_X(x)$$

as being replaced by

$$\int h(g(x)) d\mu_X(x) = \int h(y) d\mu_{g(X)}(y) = \int h(y) d\mu_Y(y) \tag{1}$$

or, when $h(v) = v$, this yields

$$\int g(x) d\mu_X(x) = \int y d\mu_Y(y)$$

where $Y \equiv g(X)$. With this set of identifications the the slightly more general identity (1) follows from the first equality, and the desired second equality is given by the special case $h(v) = v$.

5. (i) PfS, Exercise 3.3, page 45, part (a).
(ii) Suppose that $\mu$ is Lebesgue measure on the unit interval $[0,1]$ and that $(a,b) = (0,1)$ in Exercise 3.3. If $X(t,\omega) = 1_{[\omega \le t]}$, then for each $t$, $(\partial/\partial t)X(t,\omega) = 0$ almost everywhere. But $\int X(t,\omega)d\mu(\omega)$ does not differentiate to 0. Why is this not a contradiction?

**Solution:** (i) We write $h(t) \equiv \int_\Omega X(t,\omega)d\mu(\omega$. Then by linearity of the integral and the fundamental theorem of calculus it follows that

$$\begin{aligned} h(t+\epsilon) - h(t) &= \int_\Omega \left(X(t+\epsilon,\omega) - X(t,\omega)\right) d\mu(\omega) \\ &= \int_\Omega \left( \int_t^{t+\epsilon} \frac{\partial}{\partial s} X(s,\omega) ds \right) d\mu(\omega), \end{aligned}$$

and hence that

$$\frac{h(t+\epsilon) - h(t)}{\epsilon} = \int_\Omega \left( \frac{1}{\epsilon} \int_t^{t+\epsilon} \frac{\partial}{\partial s} X(s,\omega) ds \right) d\mu(\omega)$$

where the integrand $\epsilon^{-1} \int_t^{t+\epsilon} \frac{\partial}{\partial s} X(s,\omega) ds$ satisfies

$$\begin{aligned} \left|\epsilon^{-1} \int_t^{t+\epsilon} \frac{\partial}{\partial s} X(s,\omega) ds\right| &\le \frac{1}{\epsilon} \int_t^{t+\epsilon} \left|\frac{\partial}{\partial s} X(s,\omega)\right| ds \\ &\le \frac{1}{\epsilon} \int_t^{t+\epsilon} Y(\omega) ds \ \text{ if } \ [t, t+\epsilon] \subset (a,b) \\ &= Y(\omega) \in \mathcal{L}_1, \end{aligned}$$

and where

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \frac{\partial}{\partial s} X(s,\omega) ds = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} (X(t+\epsilon,\omega) - X(t,\omega)) = \frac{\partial}{\partial t} X(t,\omega)$$

for a.e. $\omega$. Thus it follows from the dominated convergence theorem that

$$\lim_{\epsilon \downarrow 0} \frac{h(t+\epsilon) - h(t)}{\epsilon} = \int_\Omega \frac{\partial}{\partial t} X(t,\omega) d\mu(\omega).$$

By using one-sided derivatives, the same argument works at the endpoints $a$ and $b$ of the interval $[a,b]$. Thus the claimed equality holds for all $t \in [a,b]$.

(ii) For $X(t,\omega) = 1_{[\omega \leq t]}$ and $\mu$ Lebesgue measure on $[0,1]$ we have

$$\int_{(0,1)} X(t,\omega) d\omega = \int_{(0,1)} 1_{[\omega \leq t]} d\omega = t,$$

so

$$\begin{aligned} \frac{\partial}{\partial t} \int_{(0,1)} X(t,\omega) d\omega &= \frac{\partial}{\partial t} t = 1 \\ &\neq 0 = \int_{(0,1)} \frac{\partial}{\partial t} X(t,\omega) d\omega. \end{aligned}$$

This does not contradict (i) above because the derivative $(\partial/\partial t) 1_{[\omega \leq t]}$ does not exist at $t = \omega$, and hence (i) does not apply.