**Statistics 521, Problem Set 3 Solutions**

Wellner; 10/18/2012

1. Exercise 2.2.1, page 28. Suppose that $(\Omega, \mathcal{A}) = (R_2, \mathcal{B}_2)$ where $\mathcal{B}_2$ denotes the $\sigma-$field generated by all open subsets of the plane. Recall that this $\sigma-$field contains all sets of the form $B \times R$ and $R \times B$ for all $B \in \mathcal{B}$ where $B_1 \times B_2 \equiv \{(r_1, r_2) : r_1 \in B_1, r_2 \in B_2\}$. now define measurable transformations $X_1(r_1, r_2) = r_1$ and $X_2(r_1, r_2) = r_2$. Then define $Z_1 \equiv \sqrt{X_1^2 + X_2^2}$ and $Z_2 \equiv \text{sign}(X_1 - X_2)$ where $\text{sign}(r) = 1, 0, -1$ according as $r$ is $> 0$, $= 0$, $< 0$. Give geometric descriptions of the $\sigma-$fields $\mathcal{F}(Z_1)$, $\mathcal{F}(Z_2)$, and $\mathcal{F}(Z_1, Z_2)$.

   **Solution:** The $\sigma-$field $\mathcal{F}(Z_1)$ is determined by circles about the origin: if $Z_1$ is known, then we know that $X_1$ and $X_2$ are on a circle with radius $Z_1$. The $\sigma-$field $\mathcal{F}(Z_2)$ is the finite $\sigma-$field generated by the three sets $L^+ \equiv \{(r_1, r_2) \in R^2 : r_1 < r_2\}$, $L \equiv \{(r_1, r_2) \in R^2 : r_1 = r_2\}$, and $L^- \equiv \{(r_1, r_2) \in R^2 : r_1 > r_2\}$. Thus if we know $Z_2$, then we know that $(X_1, X_2)$ is either above the forty-five degree line, on this line, or below it. The $\sigma-$field $\mathcal{F}(Z_1, Z_2)$ is determined by both the circles generating $\mathcal{F}(Z_1)$ and the three sets generating $\mathcal{F}(Z_2)$: if we know both $Z_1$ and $Z_2$, then we know that $(X_1, X_2)$ is either on a half-circle of radius $Z_1$ above the diagonal, on the half-circle of radius $Z_1$ where it is intersected by the diagonal, or on the half-circle of radius $Z_1$ and below the diagonal.

2. PfS, Exercise 2.2.2, page 28: Suppose that $\mathcal{C}$ is a $\overline{\pi}-$system. Suppose that $\mathcal{V}$ is a vector space of functions with:
   (i) $1_C \in \mathcal{V}$ for all $C \in \mathcal{C}$.
   (ii) If $A_n \in \mathcal{V}$ satisfy $A_n \nearrow A$, then $A \in \mathcal{V}$.
   (a) Show that $1_A \in \mathcal{V}$ for every $A \in \sigma[\mathcal{C}]$.
   (b) Show that every simple function

   $$\sum_1^m x_i 1_{A_i} \quad \text{is in} \quad \mathcal{V}$$

   whenever $m \geq 1$, $x_i \in R$, and $\sum_1^m A_i = \Omega$ with $A_i \in \sigma[\mathcal{C}]$.
   (c) Show that $\mathcal{V}$ contains all $\sigma[\mathcal{C}]-$measurable functions.

**Solution:** (a) Consider the collection of sets $\mathcal{A} = \{A \subset \Omega : 1_A \in \mathcal{V}\}$. For $C \in \mathcal{C}$ we have $1_C \in \mathcal{V}$, by hypothesis, so $C \in \mathcal{A}$, and hence $\mathcal{C} \subset \mathcal{A}$. We will show that $\mathcal{A}$ is a $\lambda-$system:
(1) First note that $\Omega \in \mathcal{A}$ since $\Omega \in \mathcal{C}$.
(2) Now suppose that $A_n \in \mathcal{A} \nearrow A$. But then $1_{A_n} \in \mathcal{V}$ with $1_{A_n} \nearrow 1_A \in \mathcal{V}$ by hypothesis, so $A \in \mathcal{A}$.
(3) Finally, suppose that $A, B \in \mathcal{A}$ with $A \subset B$. Then $1_A, 1_B \in \mathcal{V}$ and $1_{B \setminus A} = 1_B - 1_A \in \mathcal{V}$ since $\mathcal{V}$ is a vector space, and hence $B \setminus A \in \mathcal{A}$.
Thus $\mathcal{A}$ is a $\lambda-$system and $\mathcal{C} \subset \mathcal{A}$. Therefore by the $\pi - \lambda$ theorem, $\sigma[\mathcal{C}] \subset \mathcal{A}$. It follows that $1_A \in \mathcal{V}$ for all $A \in \sigma[\mathcal{C}]$.
(b) Since $\mathcal{V}$ is a vector space, it follows that all simple functions of the form $\sum_1^m x_i 1_{A_i}$ with $x_i \in R$ and $A_i \in \sigma[\mathcal{C}]$, $i = 1, \ldots, m$ are in $\mathcal{V}$.
(c) Now suppose that $X = X^+ - X^-$ is a $\sigma[\mathcal{C}]-$measurable function. Since all non-negative $\sigma[\mathcal{C}]$ measurable functions are monotone limits of simple functions and $\mathcal{V}$ is closed under monotone limits, we conclude that $X^+$, $X^- \in \mathcal{V}$, and since $\mathcal{V}$ is a vector space, this yields $X \in \mathcal{V}$.

3. PfS, Exercise 2.3.1, page 29: Let $X_1, X_2, \ldots$ denote measurable functions from $(\Omega, \mathcal{A}, \mu)$ to $(\overline{R}, \overline{\mathcal{B}})$.
(a) If $X_n \to_{a.e.} X$, then $X = \widetilde{X}$ a.e. for some measurable $\widetilde{X}$.
(b) If $X_n \to_{a.e.} X$ and $\mu$ is complete, then $X$ itself is measurable.

**Solution:** (a) Since $X_n \to_{a.e.} X$, there is a set $N \in \mathcal{A}$ with $\mu(N) = 0$ and $X_n(\omega) \to X(\omega)$ for all $\omega \in N^c$. Define $Y_n = X_n 1_{N^c}$. Then the $Y_n$'s are measurable and $Y_n(\omega) = X_n(\omega) 1_{N^c}(\omega) \to X(\omega) 1_{N^c}(\omega) \equiv \tilde{X}$ for all $\omega \in \Omega$. Since the $Y_n$'s are measurable and converge everywhere to $\tilde{X}$, the limit $\tilde{X}$ is measurable. Furthermore, $\tilde{X}(\omega) = X(\omega)$ for all $\omega \in N^c$, so $\tilde{X} = X$ a.e.

(b) From part (a) we have $[\tilde{X} \neq X] \subset N$. Since $\mu$ is complete and $\mu(N) = 0$, it follows that $[\tilde{X} \neq X] \in \mathcal{A}$ and $\mu([\tilde{X} \neq X]) = 0$. Now for any set $B \in \overline{\mathcal{B}}$ we can write

$$\begin{aligned} X^{-1}(B) &= (X^{-1}(B) \cap [X = \tilde{X}]) \cup (X^{-1}(B) \cap [X \neq \tilde{X}]) \\ &= (\tilde{X}^{-1}(B) \cap [X = \tilde{X}]) \cup C \end{aligned}$$

where $C = X^{-1}(B) \cap [X \neq \tilde{X}] \subset [X \neq \tilde{X}] \in \mathcal{A}$ with $\mu([X \neq \tilde{X}]) = 0$. By completeness of $\mu$ this yields $C \in \mathcal{A}$ and $\mu(C) = 0$. But $\tilde{X}^1(B) \in \mathcal{A}$

since $\tilde{X}$ is measurable, and $[X = \tilde{X}] \in \mathcal{A}$, and hence we conclude that $X^{-1}(B) \in \mathcal{A}$. Thus $X$ is measurable.

4. PfS, Exercise 2.3.2, page 31: (a) Show that in general $\rightarrow_\mu$ does not imply $\rightarrow_{a.e.}$.
(b) Give an example with $\mu(\Omega) = \infty$ where $\rightarrow_{a.e.}$ does not imply $\rightarrow_\mu$.

**Solution:** (a) Let $\Omega = [0,1]$, and $\mu = \lambda$ =Lebesgue measure on $[0,1]$. Now let $A_1 = [0,1/2)$, $A_2 = [1/2,1]$, $A_3 = [0,1/3)$, $A_4 = [1/3,2/3)$, $A_5 = [2/3,1]$, .... Now let $X_n(\omega) = 1_{A_n}(\omega)$ for $n = 1,2,\ldots$, and let $X(\omega) = 0$. Now $X_n \rightarrow_\mu X = 0$ if $\mu([|X_n| > \epsilon]) \rightarrow 0$ as $n \rightarrow \infty$ for every $\epsilon > 0$. In this case, for each $\epsilon \in (0,1)$ $\mu([|X_n| > \epsilon]) = \mu(A_n) \rightarrow 0$ as $n \rightarrow \infty$, so $X_n \rightarrow_\mu X = 0$. However, $X_n \rightarrow_{a.e.} X = 0$ iff $\mu([|X_n| > \epsilon] \text{ i.o.}) = 0$ for every $\epsilon > 0$. But for any $\epsilon \in (0,1)$ we have $\{\omega \in \Omega : |X_n(\omega)| > \epsilon \ \text{ i.o.}\} = [0,1]$ by construction of the intervals $A_n$, and hence $\mu([|X_n| > \epsilon] \text{ i.o.}) = 1$. Hence $X_n \not\rightarrow_{a.e.} X$.
(b) Let $\Omega = [0,\infty)$ with $\mu = \lambda$ =Lebesgue measure. Set $X_n(\omega) = 1_{[n,n+1)}(\omega)$ for $n = 1,2,\ldots$, and $X(\omega) = 0$. Now $X_n \rightarrow_{a.e.} X$ and in fact, since $X_n(\omega) = 0$ for all $n > \omega$, $X_n(\omega) \rightarrow X(\omega) = 0$ for every $\omega \in \Omega$. But $X_n \not\rightarrow_\mu 0$ because, for each $\epsilon \in (0,1)$,

$$\mu([|X_n| > \epsilon]) = 1 \not\rightarrow 0\,.$$

5. PfS, Exercise 2.3.3, page 32: show that $X_n \rightarrow_\mu X$ if and only if $X_n - X_m \rightarrow_\mu 0$.

**Solution:** First suppose that $X_n \rightarrow_\mu X$. Then for every $\epsilon > 0$ we have

$$\mu([|X_m - X_n| > \epsilon]) \le \mu([|X_m - X| > \epsilon/2]) + \mu([|X_n - X| > \epsilon/2]) \rightarrow 0$$

as $m, n \rightarrow \infty$; i.e. $X_n - X_m \rightarrow_\mu 0$.

Now suppose that $X_n - X_m \rightarrow_\mu 0$. First, choose a subsequence $n_k$ increasing so that

$$\mu([|X_{n_k} - X_l| > 2^{-k}]) < 2^{-k} \qquad \text{for all} \;\; l > n_k.$$

Let $A_k \equiv [|X_{n_k} - X_{n_{k+1}}| > 2^{-k}]$. Set $B_m \equiv \cup_{k=m}^\infty A_k$, and note that

$$\mu(B_m) \le \sum_{k=m}^\infty \mu(A_k) < \sum_{m}^\infty 2^{-k} = 2^{-(m-1)}\,.$$

On $B_m^c = \cap_{k=m}^{\infty} A_k^c$ we have $|X_{n_k} - X_{n_{k+1}}| \le 2^{-k}$ for all $k \ge m$. Moreover, for $n_i > n_j > m$ it follows that

$$|X_{n_i}(\omega) - X_{n_j}(\omega)| \le \sum_{k=j}^{\infty} |X_{n_k}(\omega) - X_{n_{k+1}}(\omega)| < 2^{-(j-1)}$$

for $\omega \in B_m^c$, and this implies that $X_{n_k}(\omega) \to X(\omega)$ for all $\omega \in C \equiv \cup_1^{\infty} B_m^c$ with

$$\mu(C^c) = \mu(\cap_1^{\infty} B_m) \le \limsup \mu(B_m) \le \lim 2^{-(m-1)} = 0\,.$$

Define $X(\omega) = 0$ for $\omega \in C^c$; then $X$ is measurable, and we have

$$\mu([|X_n - X| \ge \epsilon]) \le \mu([|X_n - X_{n_k}| \ge \epsilon/2]) + \mu([|X_{n_k} - X| \ge \epsilon/2]) \to 0$$

as $n \ge n_k \to \infty$.