Statistics 521, Problem Set 10, Solutions

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1. Show that if X_n is any sequence of random variables, there are constants $c_n \to \infty$ so that $X_n / c_n \to_{a.s.} 0$.

   Solution: Define $F_n(x) \equiv P(|X_n| \le x)$, the distribution function of $|X_n|$. Set $b_n = F_n^{-1}(1 - n^{-2})$ for $n = 1, 2, \ldots$, and let $\{a_n\}$ be any sequence with $a_n \to \infty$. Let $c_n = a_n b_n$. Then, for any $\epsilon > 0$ we have $\epsilon a_n \ge 1$ for $n \ge N_\epsilon$ and hence, using $F_n \circ F_n^{-1}(t) \ge t$ for all $0 < t < 1$,

$$\begin{aligned} P(|X_n| > \epsilon c_n) &= P(|X_n| > \epsilon a_n b_n) \\ &\le P(|X_n| > b_n) \\ &= 1 - F_n(b_n) = 1 - F_n(F_n^{-1}(1 - n^{-2})) \\ &\le n^{-2}, \quad n \ge N_\epsilon. \end{aligned}$$

   Hence by the first Borel-Cantelli lemma, $P(|X_n| > \epsilon c_n \ \text{ i.o.}) = 0$ for every $\epsilon > 0$; that is, $X_n / c_n \to_{a.s.} 0$.

2. Show that if $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$, then $P(A_n \text{ i.o.}) = 0$.

   Solution: If the A_n's are decreasing, this is fairly easy: then we can write

$$\begin{aligned} \cup_{k=n}^{\infty} A_k &= \cup_{k=n}^{\infty} A_k \cap A_{k+1}^c + \cup_{k=n}^{\infty} A_k \cap A_{k+1} \\ &= \cup_{k=n}^{\infty} A_k \cap A_{k+1}^c + A_{n+1}, \end{aligned}$$

   and this yields

$$\begin{aligned} P(A_n \text{ i.o.}) &= \lim_{n \to \infty} P(\cup_{k=n}^{\infty} A_k) \\ &\le \lim_{n \to \infty} \left\{ P(\cup_{k=n}^{\infty} A_k \cap A_{k+1}^c) + P(A_{n+1}) \right\} \\ &= P(A_n \cap A_{n+1}^c \text{ i.o.}) + 0 = 0\,. \end{aligned}$$

   On the other hand, if the A_n's are not decreasing, this is a bit trickier. Now use the usual disjointfication procedure: let $B_1 = A_1$, $B_2 =$

$A_n^c A_{n+1}$, ... , $B_k = A_n^c A_{n+1}^c \cdots A_{n+k-2}^c A_{n+k-1}, \cdots$. Then

$$\begin{aligned} \cup_{k=n}^{\infty} A_k &= \sum_{k=1}^{\infty} B_k \\ &= A_n + \cup_{k=n} A_{k+1} \cap_{j=n}^{\infty} A_j^c \\ &\subset A_n + \cup_{k=n} A_{k+1} A_k^c\,. \end{aligned}$$

Since the left side is decreasing in n, this implies that for each fixed n we have

$$[A_n \text{ i.o.}] \subset A_n + [A_{n+1} \cap A_n^c \text{ i.o.}]\,.$$

But $[A_{n+1} \cap A_n^c \text{ i.o.}] = [A_n \cap A_{n+1}^c \text{ i.o.}]$, so

$$[A_n \text{ i.o.}] \subset A_n + [A_n \cap A_{n+1}^c \text{ i.o.}]$$

for every fixed n. Thus

$$P(A_n \text{ i.o.}) \leq P(A_n) + P(A_n \cap A_{n+1}^c \text{ i.o.}) = P(A_n) + 0$$

by the first Borel-Cantelli lemma. Since $P(A_n) \to 0$, it follows that $P(A_n \text{ i.o.}) = 0$.

3. Let $X_1, X_2, \ldots$ be independent. Show that $\sup X_n < \infty$ almost surely if and only if $\sum_n P(X_n > M) < \infty$ for some $M < \infty$.

   Solution: Suppose that $\sum_n P(X_n > M) < \infty$ for some $M < \infty$. Then by the first Borel-Cantelli lemma, $P(X_n > M \text{ i.o.}) = 0$; i.e. for $n \geq N_\omega$ we have $X_n(\omega) \leq M$. Thus

   $$\sup_n X_n(\omega) \leq \left(\max_{1 \leq k < N_\omega} X_k \right) \vee M \; < \; \infty\,.$$

   Now suppose that $\sup X_n < \infty$ almost surely. If $\sum_n P(X_n > M) = \infty$ for every $M < \infty$, then, by the second Borel-Cantelli lemma, $P(X_n > M \text{ i.o.}) = 1$ for every M; i.e. $\limsup_{n\to\infty} X_n \geq M$ a.s. for every $M > 0$, and this implies, by taking a sequence $M_k \nearrow \infty$, that $\limsup_{n\to\infty} X_n = \infty$ a.s., which contradicts $\sup X_n < \infty$ almost surely. We therefore conclude that $\sum_n P(X_n > M) < \infty$ for some $M < \infty$.

4. Let $X_1, X_2, \ldots$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show that: (i) $X_n \to_p 0$ if and only if $p_n \to 0$, and $X_n \to_{a.s.} 0$ if and only if $\sum_n p_n < \infty$.

   Solution: Here, for any $0 < \epsilon < 1$, $P(|X_n| > \epsilon) = P(X_n = 1) = p_n$. Thus $X_n \to_p 0$ if and only if $p_n \to 0$.
   By the first and second Borel-Cantelli lemmas and the first part, for $0 < \epsilon < 1$ we have

$$P(|X_n| > \epsilon \text{ i.o.}) = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \quad \text{according as} \quad \sum_{n=1}^{\infty} p_n \left\{ \begin{array}{c} < \infty \\ = \infty \end{array} \right\} .$$

   It follows that $X_n \to_{a.s.} 0$ if and only if $\sum_{n=1}^{\infty} p_n < \infty$.

5. Suppose that $X_1, X_2, \ldots$ are independent with $P(X_n > x) = x^{-r}$ for all $x \ge 1$ and $n = 1, 2, \ldots$ with $r > 0$. Show that $\limsup_{n\to\infty} (\log X_n)/\log n = c$ almost surely for some number c, and find c.

   Solution: Let $c > 0$. Then

$$P(\log X_n > c \log n) = P(X_n > n^c) = n^{-5r}.$$

   Hence by the first and second Borel-Cantelli lemmas

$$P(\log X_n > c \log n \ \text{ i.o.}) = \left\{ \begin{array}{ll} 0 & \text{if } \ c > 1/r \\ 1 & \text{if } \ c \le 1/r \end{array} \right. .$$

   Hence

$$\limsup_{n\to\infty} (\log X_n)/(\log n) = 1/r \qquad \text{almost surely} .$$

6. (a) Suppose that $X_1, X_2, \ldots$ are random variables with mean 0, $EX_j^2 = 1$, and $E(X_i X_j) = 0$ for all $i \ne j$, and let $S_n \equiv X_1 + \cdots + X_n$. Show that $S_n/n^\alpha \to_{a.s.} 0$ for any $\alpha > 1$.
   (b) Suppose that $X_1, X_2, \ldots$ are random variables with mean 0, $E(X_i X_j) = 0$ for all $i \ne j$, and $\sup_j EX_j^2 < \infty$. Show that $S_n/n^\alpha \to_p 0$ for any $\alpha > 1/2$.

   Solution: (a) Let $\epsilon > 0$. Then by Chebychev's inequality

$$P(|S_n|/n^\alpha > \epsilon) \le \epsilon^{-2} n^{-2\alpha} \sum_{i=1}^{n} Var(X_i) = \epsilon^{-2} n^{-(2\alpha - 1)}$$

since $Var(X_i) = 0$ and $E(X_i X_j) = 0$ for $i \neq j$. Thus it follows that

$$\sum_{n=1}^{\infty} P(|S_n|/n^\alpha > \epsilon) \leq \epsilon^{-2} \sum_{n=1}^{\infty} n^{-(2\alpha-1)} < \infty$$

for $\alpha > 1$ since this implies that $2\alpha - 1 > 1$. Thus by the first Borel-Cantelli lemma $P(|S_n|/n^\alpha > \epsilon \ i.o.) = 0$. This implies that $\limsup_{n\to\infty} n^{-\alpha}|S_n| \leq \epsilon$ a.s. for each $\epsilon > 0$. Thus we conclude that

$$\limsup_{n\to\infty} n^{-\alpha}|S_n| = 0;$$

i.e $n^{-\alpha} S_n \to_{a.s.} 0$.
(b) Let $\epsilon > 0$. By Chebychev's inequality again

$$\begin{aligned} P(|S_n|/n^\alpha > \epsilon) &\leq \epsilon^{-2} n^{-2\alpha} \sum_{i=1}^{n} Var(X_i) \leq M\epsilon^{-2} n^{-(2\alpha-1)} \\ &\to 0 \end{aligned}$$

since $\alpha > 1/2$; here $\sup_{i\geq 1} Var(X_i) = \sup_{i\geq 1} E(X_i^2) \equiv M < \infty$. We conclude that $S_n/n^\alpha \to_p 0$ if $\alpha > 1/2$.