Statistics 521, Midterm Exam Solutions

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1. (24 points). **Define** *three* of the following five terms:
   (a) Convergence in measure of a sequence of measurable functions $\{X_n\}$ defined on the measure space $(\Omega, \mathcal{A}, \mu)$.
   (b) Convergence in distribution of a sequence of random variables $\{X_n\}$.
   (c) $\limsup A_n$ for a sequence of events $\{A_n\}$, and give the common (intuitive) abbreviation for this set.
   (d) A *simple function* defined on a measurable space $(\Omega, \mathcal{A})$.
   (e) The Lebesgue integral $\int X d\mu$ of a (real-valued) measurable function X defined on a measure space $(\Omega, \mathcal{A}, \mu)$.

   Solution: See PfS, chapters 1-3.

2. (20 points). Give careful **statements** of *two* of the following four theorems:
   (a) The dominated convergence theorem.
   (b) A theorem relating convergence in measure ($\rightarrow_\mu$) to convergence almost everywhere ($\rightarrow_{a.e.}$).
   (c) The C_r−inequality.
   (d) The Caratheodory extension theorem.
   (e) The "theorem of the unconscious statistician".

   Solution: See PfS, chapters 1-3.

3. (30 points).
   (a) Suppose that X is a non-negative measurable function on a measurable space $(\Omega, \mathcal{A})$. Give an explicit sequence of simple functions X_n satisfying $X_n \nearrow X$.
   (b) Now suppose that $(\Omega, \mathcal{A}) = ((0,1), \mathcal{B}_{(0,1)})$, and that we give this measurable space the Lebesgue measure λ, which we call P since it is a probability measure on this $(\Omega, \mathcal{A})$. Suppose that $X(\omega) = \omega^{-2/3}$ for $\omega \in (0,1)$.
   (b-1) For the simple functions X_n as given in (a), evaluate

$$\lim_{n\to\infty} \int X_n dP = \lim_{n\to\infty} E(X_n).$$

   (b-2) Find the (induced) distribution function $F = F_X$ of X on $\mathbb{R}$.

   Solution: (a) From section 2.2 (and (2.2.10) in particular), for $X \geq 0$ we can take

$$X_n \equiv \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1\{\frac{k-1}{2^n} \leq X < \frac{k}{2^n}\} + n 1_{[X \geq n]}.$$

   Then $X_n(\omega) \nearrow X(\omega)$ for every ω.
   (b1) By the monotone convergence theorem,

$$E(X_n) \to E(X) = \int_0^1 \omega^{-2/3} d\omega = \frac{\omega^{1/3}}{1/3}\Big|_0^1 = 3.$$

(b2) By definition

$$\begin{aligned} F(x) &= P(X \le x) = P(\{\omega : \omega^{-2/3} \le x\}) = P(\{\omega : \omega \ge x^{-3/2}\}) \\ &= \begin{cases} 1 - x^{-3/2}, & \text{for } x \ge 1, \\ 0, & \text{for } x \le 1. \end{cases} \end{aligned}$$

Also note that

$$E(X) = \int_0^\infty (1 - F(x))dx = \int_0^1 1dx + \int_1^\infty x^{-3/2}dx = 1 + 2 = 3.$$

4. (32 points).
Let $(\Omega, \mathcal{A}, \mu) = ([0,1], \mathcal{B}, P)$ with $P = \lambda$ being Lebesgue measure (on $[0,1]$). For $\omega \in [0,1]$, define measurable functions X_n by $X_n(\omega) = \omega 1_{(n^{-2},1]} + n 1_{[0,n^{-2}]}$.
(a) Does $X_n \to_{a.s.}$ "some" X? (b) Does $X_n \to_p$ "some" X? (c) Does $X_n \to_d$ "some" X? If so, what is the distribution function F of X?
(d) Let $g(x) = (1/2)x1_{[0,1/2)}(x) + ((1/2)x + 1/2)1_{[1/2,1]}(x)$. Does $g(X_n)) \to_p$ something? If so, what is something?
(e) Compute EX_n^r for $r > 0$ and $n = 1, 2, \ldots$.
(f) For what values of $r > 0$ does $EX_n^r \to$ something finite?
(g) For what values of $r > 0$ does $EX_n^r \to EX^r$?

Solution:
(a) $X_n(\omega) \to \tilde{X}(\omega) = \omega 1_{(0,1]}(\omega) + \infty \cdot 1_{\{0\}}(\omega)$, and since $\tilde{X}(\omega) = \omega \equiv X(\omega)$ a.s. (since $P([X = \tilde{X}]) = P(\{\omega : \ \omega \in (0,1]\}) = 1$), it follows that $X_n \to_{a.s.} X$.
(b) Yes, since $X_n \to_{a.s.} X$ implies $X_n \to_p X$ (for finite measure spaces).
(c) Yes, since $X_n \to_p X$ implies $X_n \to_d X$. In this case

$$F(x) = P(X \le x) = 0 \cdot 1_{(-\infty,0)]}(x) + x \cdot 1_{[0,1]}(x) + 1_{[1,\infty)}(x),$$

the distribution function of a $U(0,1)$ random variable.
(d) This function g is continuous a.s. P_X since $P(X = 1/2) = 0$. Thus $g(X_n) \to_p g(X)$ by the continuous mapping theorem.
(e) Now

$$\begin{aligned} E(X_n^r) &= \int_0^1 \left\{\omega^r 1_{(n^{-2},1]}(\omega) + n^r 1_{[0,n^{-2})}(\omega)\right\} d\omega \\ &= (r+1)^{-1}\{1^{r+1} - (n^{-2})^{r+1}\} + n^r \cdot n^{-2}. \end{aligned}$$

(f) From the computation in (e) we see that

$$E(X_n^r) \to \frac{1}{r+1} + 1_{\{2\}}(r) + \infty \cdot 1_{(2,\infty)}(r) < \infty \ \text{ if } r \le 2.$$

(g) Since $EX^r = \int_0^1 \omega^r d\omega = (r+1)^{-1}$, it follows that $EX_n^r \to EX^r$ if $r < 2$.