Statistics 521, Problem Set 8

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Reading: Shorack, PfS, Chapter 5, pages 87 - 101.
Shorack, PfS, Chapter 7, pages 123 - 129.

Due: Wednesday, November 21, 2012.

1. PfS Exercise 6.4.3, page 114. Prove just the parts of these formulas involving F, not the parts involving F^{-1}. You may also use Fubini's theorem directly. That is, show that:
(i) If $X \ge 0$ has d.f. F, then $\int_0^\infty P(X > x)dx = E(X) = \int_0^\infty (1 - F(x))dx$.
(ii) If $E|X| < \infty$ then $E(X) = -\int_{-\infty}^0 F(x)dx + \int_0^\infty (1 - F(x))dx$.
(iii) Let $r > 0$. If $X \ge 0$, then $\int_0^\infty P(X^r > x) = E(X^r) = \int_0^\infty r x^{r-1}(1 - F(x))dx$.

2. Prove the two formulas in (17), PfS page 113: if $X \ge 0$ is integer valued, then $E(X) = \sum_{k=1}^\infty P(X \ge k)$ and $E(X^2) = \sum_{k=1}^\infty (2k-1)P(X \ge k)$.

3. PfS, Exercise 5.1.4, page 91.
Let $\mathcal{X} = [0, 1]$, $\mathcal{Y} = (1, \infty)$ both equipped with the Borel sets and Lebesgue measure. Let $f(x, y) = e^{-xy} - 2e^{-2xy}$. Show that
(i) $\int_0^1 (\int_1^\infty f(x, y)dy)dx = \int_0^1 x^{-1}(e^{-x} - e^{-2x})dx$ exists and is > 0.
(ii) $\int_1^\infty (\int_0^1 f(x, y)dx)dy = \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy$ exists and is < 0.

4. Show that $f(x, y) = e^{-xy}\sin(x)$ is integrable with respect on Lebesgue measure on R^2 in the strip $0 < x < a$, $0 < y$. Perform the double integral in the two different orders to find that

$$\int_0^a \frac{\sin(x)}{x}dx = \frac{\pi}{2} - \cos(a)\int_0^\infty \frac{e^{-ay}}{1+y^2}dy - \sin(a)\int_0^\infty \frac{ye^{-ay}}{1+y^2}dy\,.$$

Use the inequality $1 + y^2 \ge 1$ to obtain the bound

$$\left|\int_0^a \frac{\sin(x)}{x}dx - \frac{\pi}{2}\right| \le \frac{2}{a}\,.$$

Letting $a \to \infty$ yields $\int_0^\infty x^{-1}\sin(x)dx = \pi/2$.

5. **Optional bonus problem:** Suppose that $\phi : (0,1) \to R^+$ satisfies:
(i) $\phi(1) = 0$,
(ii) $\phi' < 0$,
(iii) $\phi'' > 0$.
(iv) $\phi(u) \to \infty$ as $u \downarrow 0$.
For example: (a) $\phi(u) = (u^{-\theta} - 1)/\theta$, $0 < \theta < \infty$; (b) $\phi(u) = (-\log u)^{1/\theta}$, $0 < \theta < 1$; (c) $\phi(u) = \log(\frac{1-\theta}{1-\theta^u})$, $0 < \theta < \infty$, $\theta \neq 1$.
Define $C : (0,1] \times (0,1] \to [0,1]$ by $C(u,v) = \phi^{-1}(\phi(u) + \phi(v))$.
A. Show that C is the joint distribution function of a pair of random variables (U,V) which have the property that the marginal distributions of U and V are Uniform$(0,1)$.
B. Show that under (i) - (iv) (U,V) have a joint density function $c(u,v)$ and calculate it; i.e. show that $C(u,v) \ll$Lebesgue measure on $[0,1]^2$.
C. What if $\lim_{u\to 0} \phi(u) = \phi(0) < \infty$? Does (U,V) have a joint density function? Show that if $\lim_{u\to 0} \phi(u) = \phi(0) < \infty$, then C is still a distribution function, but with one component C_{ac} with a density function c_{ac} ("ac" for "absolutely continuous with respect to Lebesgue measure λ_2 on R^2), and a component which is singular with respect to λ_2. Show that the singular component has probability $p = -\frac{\phi(0)}{\phi'(0)}\}$ on the curve given by: $\phi(u) + \phi(v) = \phi(0)$.
Notes: the d.f. C defined above is an *Archimedean copula*: a distribution function on the unit square with uniform$(0,1)$ marginal distributions.