Statistics 521, Problem Set 10

Wellner; 11/28/2007

Reading: Shorack, PfS, Chapter 8, pages 147 - 174.

Reminder: Final exam, 2:30 - 4:20, Wednesday, December 12.

Due: Wednesday, December 5, 2012.

1. Show that if X_n is any sequence of random variables, there are constants $c_n \to \infty$ so that $X_n/c_n \to_{a.s.} 0$.

2. Show that if $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$, then $P(A_n \text{i.o.}) = 0$.

3. Let $X_1, X_2, \ldots$ be independent. Show that $\sup X_n < \infty$ almost surely if and only if $\sum_n P(X_n > M) < \infty$ for some $M < \infty$.

4. Let $X_1, X_2, \ldots$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show that: (i) $X_n \to_p 0$ if and only if $p_n \to 0$, and $X_n \to_{a.s.} 0$ if and only if $\sum_n p_n < \infty$.

5. Suppose that $X_1, X_2, \ldots$ are independent with $P(X_n > x) = x^{-r}$ for all $x \ge 1$ and $n = 1, 2, \ldots$ with $r > 0$. Show that $\limsup_{n\to\infty} (\log X_n)/\log n = c$ almost surely for some number c, and find c.

6. (a) Suppose that $X_1, X_2, \ldots$ are random variables with mean 0, $EX_j^2 = 1$, and $E(X_i X_j) = 0$ for all $i \ne j$, and let $S_n \equiv X_1 + \cdots + X_n$. Show that $S_n/n^\alpha \to_{a.s.} 0$ for any $\alpha > 1$.
   (b) Suppose that $X_1, X_2, \ldots$ are random variables with mean 0, $E(X_i X_j) = 0$ for all $i \ne j$, and $\sup_j EX_j^2 < \infty$. Show that $S_n/n^\alpha \to_p 0$ for any $\alpha > 1/2$.

7. **Optional bonus problem:** Suppose $U(\omega) = \omega$ for

$$(\Omega, \mathcal{A}, P) = ((0,1], \mathcal{B}_{(0,1]}, \lambda)$$

where λ is Lebesgue measure (restricted to $(0,1]$). Thus $U \sim \text{Uniform}(0,1)$. Define

$$T(\omega) = \begin{cases} 2\omega, & 0 < \omega \le 1/2, \\ 2\omega - 1, & 1/2 < \omega \le 1, \end{cases} \qquad X_1(\omega) = \begin{cases} 0, & 0 < \omega \le 1/2, \\ 1, & 1/2 < \omega \le 1, \end{cases}$$

and, for $i \ge 2$,

$$X_i(\omega) = X_1(T^{i-1}\omega).$$

It follows that

$$\sum_{i=1}^{n} \frac{X_i(\omega)}{2^i} < \omega \le \sum_{i=1}^{n} \frac{X_i(\omega)}{2^i} + \frac{1}{2^n}$$

and the X_i's give the diadic (non-terminating expansion) representation of U:

$$U(\omega) = \sum_{i=1}^{\infty} \frac{X_i(\omega)}{2^i}.$$

Show that $X_1, X_2, \ldots$ are independent random variables.
[Hint: see Billingsley, *Probability Theory and Measure*, pages 1-5 and A31, page 572.]