

Statistics 521, Problem Set 9 Solutions

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1. PfS, Exercise 5.3.1, page 94.

Prove that:

- (a) A function $\underline{X} = (X_1, X_2, \dots) : \Omega \rightarrow R^\infty$ is $\mathcal{B}_\infty - \mathcal{A}$ -measurable if and only if each X_n is $\mathcal{B} - \mathcal{A}$ -measurable.
(b) If $\underline{X} = (X_1, X_2, \dots)$ is $\mathcal{B}^\infty - \mathcal{A}$ -measurable and if (i_1, i_2, \dots) is an arbitrary sequence of integers, then $\underline{Y} \equiv (X_{i_1}, X_{i_2}, \dots)$ is $\mathcal{B}^\infty - \mathcal{A}$ -measurable.

Solution: (a) First suppose that $\underline{X} = (X_1, X_2, \dots) : \Omega \rightarrow R^\infty$ is $\mathcal{B}_\infty - \mathcal{A}$ -measurable. Fix $n \geq 1$. Then for any Borel set $B \in \mathcal{B}$ we have $X_n^{-1}(B) = \underline{X}^{-1}(R \times \dots \times R \times B \times R \times \dots) \in \mathcal{A}$, and hence X_n is $\mathcal{B} - \mathcal{A}$ -measurable.

Now suppose that X_n is $\mathcal{B} - \mathcal{A}$ -measurable for each n . Since $\mathcal{B}^\infty = \sigma[\mathcal{C}_I]$, it suffices to show that $\underline{X}^{-1}(I) \in \mathcal{A}$ for any finite-dimensional rectangle $I = I_1 \times I_2 \times \dots \times I_n \times R \times \dots$ with each $I_j \in \mathcal{B}$. But we have

$$\begin{aligned} \underline{X}^{-1}(I) &= \underline{X}^{-1}(I_1 \times I_2 \times \dots \times I_n \times R \times \dots) \\ &= \bigcap_{k=1}^n X_k^{-1}(I_k) \cap \bigcap_{k=n+1}^\infty X_k^{-1}(R) \\ &= \bigcap_{k=1}^n X_k^{-1}(I_k) \cap \Omega \\ &= \bigcap_{k=1}^n X_k^{-1}(I_k) \\ &\in \mathcal{A} \end{aligned}$$

since $X_k^{-1}(I_k) \in \mathcal{A}$ for each k by hypothesis.

- (b) From the “only if” part of (a), $\underline{X} = (X_1, X_2, \dots)$ is $\mathcal{B}^\infty - \mathcal{A}$ -measurable implies that X_k is $\mathcal{B} - \mathcal{A}$ -measurable for each k , and hence X_{i_k} is $\mathcal{B} - \mathcal{A}$ -measurable for each k . Then by the “if” part of (a), $\underline{Y} \equiv (X_{i_1}, X_{i_2}, \dots)$ is $\mathcal{B}^\infty - \mathcal{A}$ -measurable.

2. PfS, Exercise 8.1.1, page 166.

- (a) Show that $P(AB) = P(A)P(B)$ if and only if $\{\emptyset, A, A^c, \Omega\}$ and $\{\emptyset, B, B^c, \Omega\}$ are independent σ -fields.

(b) Show that A_1, \dots, A_n are independent if and only if (for each $k = 1, \dots, n$,

$$P(A_{i_1} \dots A_{i_k}) = \prod_{j=1}^k P(A_{i_j}) \quad \text{whenever } 1 \leq i_1 < \dots < i_k \leq n.$$

Solution: We will prove (b) first.

(b) Since A_1, \dots, A_n are independent if and only if the random variables $1_{A_1}, \dots, 1_{A_n}$ are independent, if and only if the σ -fields $\mathcal{F}(1_{A_1}), \dots, \mathcal{F}(1_{A_n})$ are independent, and of course these are just the σ -fields

$$\mathcal{A}_1 \equiv \{\emptyset, A_1, A_1^c, \Omega\}, \dots, \mathcal{A}_n \equiv \{\emptyset, A_n, A_n^c, \Omega\}.$$

It remains only to show that the σ -fields $\mathcal{A}_1, \dots, \mathcal{A}_n$ are independent if and only if for each $k = 1, \dots, n$

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = \prod_{j=1}^k P(A_{i_j}) \quad (1)$$

whenever $1 \leq i_1 < \dots < i_k \leq n$. First suppose that the σ -fields $\mathcal{A}_1, \dots, \mathcal{A}_n$ are independent. Then by taking $\Omega \in \{\emptyset, A_j, A_j^c, \Omega\}$ for $j \in \{i_1, \dots, i_k\}^c$, we have, with $B_j = A_{i_m}$ if $j = i_m$, $B_j = \Omega$ if $j \in \{i_1, \dots, i_k\}^c$,

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = P(\cap_{j=1}^n B_j) = \prod_{j=1}^n P(B_j) = P(A_{i_1}) \dots P(A_{i_k})$$

since $P(B_j) = P(\Omega) = 1$ for $j \in \{i_1, \dots, i_k\}^c$; i.e. (1) holds. Now suppose that (1) holds. In particular, this implies that

$$P(B_1 \cap \dots \cap B_n) = P(B_1) \dots P(B_n) \quad (2)$$

where each $B_i \in \{\emptyset, A_i, \Omega\}$, $i = 1, \dots, n$, since both sides are 0 if any $B_j = \emptyset$, and if $B_j = A_j$ for $j \in \{j_1, \dots, j_m\} \subset \{1, \dots, n\}$, $B_j = \Omega$ for $j \in \{j_1, \dots, j_m\}^c \subset \{1, \dots, n\}$, then (2) reduces to (1). Now consider replacing B_1 by the other remaining element of \mathcal{A}_1 , A_1^c : if we replace B_1 by A_1^c , then since $\Omega = A_1 + A_1^c$ it follows that $\Omega \cap \cap_{k=2}^n A_k = \cap_{k=1}^n A_k + A_1^c \cap_{k=2}^n A_k$ and hence the left side of (2) becomes

$$\begin{aligned} P(\cap_{k=2}^n A_k) - P(A_1 \cap \cap_{k=2}^n A_k) &= P(B_2) \dots P(B_n) - P(A_1)P(B_2) \dots P(B_n) \\ &= (1 - P(A_1))P(B_2) \dots P(B_n) \\ &= P(A_1^c)P(B_2) \dots P(B_n); \end{aligned}$$

thus we have proved that

$$P(C_1 \cap B_2 \dots \cap B_n) = P(C_1)P(B_2) \cdots P(B_n) \quad (3)$$

for $C_1 \in \mathcal{A}_1$, and $B_j \in \{\emptyset, A_j, \Omega\}$ for $j = 2, \dots, n$. This is the first step of an induction argument. Now suppose that for some $k \in \{1, \dots, n\}$.

$$P(C_1 \cdots C_{k-1} \cap B_k \cdots B_n) = P(C_1) \cdots P(C_{k-1})P(B_k) \cdots P(B_n) \quad (4)$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \dots, k-1$, $B_i \in \{\emptyset, A_i, \Omega\}$, $i = k, \dots, n$. Since $\Omega = A_k + A_k^c$ it follows that

$$\Omega \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^n B_j = A_k \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^n B_j + A_k^c \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^n B_j.$$

Thus upon replacing B_k by A_k^c on the left side of (4), we see that we have, since both $\Omega, A_k \in \{\emptyset, A_k, \Omega\}$,

$$\begin{aligned} & P(\cap_{j=1}^{k-1} C_j \cap A_k^c \cap \cap_{j=k+1}^n B_j) \\ &= P(\cap_{j=1}^{k-1} C_j \cap \Omega \cap \cap_{j=k+1}^n B_j) - P(\cap_{j=1}^{k-1} C_j \cap A_k \cap \cap_{j=k+1}^n B_j) \\ &= \prod_{j=1}^{k-1} P(C_j)P(\Omega) \prod_{j=k+1}^n P(B_j) - \prod_{j=1}^{k-1} P(C_j)P(A_k) \prod_{j=k+1}^n P(B_j) \\ &= \prod_{j=1}^{k-1} P(C_j) \cdot (1 - P(A_k)) \cdot \prod_{j=k+1}^n P(B_j) \\ &= \prod_{j=1}^{k-1} P(C_j) \cdot P(A_k^c) \cdot \prod_{j=k+1}^n P(B_j). \end{aligned}$$

Hence we have proved that (4) implies that

$$P(C_1 \cdots C_k B_{k+1} \cdots B_n) = P(C_1) \cdots P(C_k)P(B_{k+1}) \cdots P(B_n) \quad (5)$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \dots, k$, $B_i \in \{\emptyset, A_i, \Omega\}$, $i = k+1, \dots, n$. It then follows by induction that

$$P(C_1 \cdots C_n) = P(C_1) \cdots P(C_n) \quad (6)$$

for all $C_i \in \mathcal{A}_i$, $i = 1, \dots, n$; i.e. the σ -fields \mathcal{A}_i , $i = 1, \dots, n$ are independent.

(a) This follows immediately from (a) with $n = 2$.

3. PfS, Exercise 8.1.3, page 168.

If $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent classes of events and if each \mathcal{C}_i is a $\bar{\pi}$ -system, then $\sigma[\mathcal{C}_1], \dots, \sigma[\mathcal{C}_n]$ are independent σ -fields.

Solution: First Proof: Let \mathcal{C} be the class of “product” sets

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \dots \times \mathcal{C}_n = \{C_1 \times \dots \times C_n : C_i \in \mathcal{C}_i, i = 1, \dots, n\}.$$

Note that \mathcal{C} is a $\bar{\pi}$ -system for the product space $\Omega \times \dots \times \Omega$: if $A = A_1 \times \dots \times A_n \in \mathcal{C}$ and $B = B_1 \times \dots \times B_n \in \mathcal{C}$, then $A \cap B = A_1 \cap B_1 \times \dots \times A_n \cap B_n \in \mathcal{C}$ since each \mathcal{C}_i is a $\bar{\pi}$ -system. Now define two measures on $(\Omega^n, \mathcal{A}^n) = (\Omega \times \dots \times \Omega, \mathcal{A} \times \dots \times \mathcal{A})$:

$$Q_1(A_1, \dots, A_n) = P(\cap_{i=1}^n A_i); \quad Q_2(A_1, \dots, A_n) = \prod_{i=1}^n P(A_i).$$

Then $Q_1 = Q_2$ on \mathcal{C} by hypothesis. Hence by Dynkin’s lemma, $Q_1 = Q_2$ on $\sigma[\mathcal{C}] = \sigma[\mathcal{C}_1] \times \dots \times \sigma[\mathcal{C}_n]$, and hence $\sigma[\mathcal{C}_1], \dots, \sigma[\mathcal{C}_n]$ are independent σ -fields.

Second Proof: Since $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent, if $C_i \in \mathcal{C}_i$ for $i = 1, \dots, n$, then

$$P(C_1 \cap \dots \cap C_n) = \prod_{j=1}^n P(C_j).$$

Fix $C_2 \in \mathcal{C}_2, \dots, C_n \in \mathcal{C}_n$, and define the collection \mathcal{D} by

$$\mathcal{D} = \{D \in \mathcal{A} : P(D \cap C_2 \cap \dots \cap C_n) = P(D) \prod_{j=2}^n P(C_j)\}.$$

Then we clearly have $\mathcal{C}_1 \subset \mathcal{D}$. Moreover, \mathcal{D} is a λ -system since:

(a) $\Omega \in \mathcal{C}_1 \subset \mathcal{D}$.

(b) If $A, B \in \mathcal{D}$ with $B \subset A$, then

$$P(A \cap C_2 \cap \dots \cap C_n) = P(A) \prod_{j=2}^n P(C_j), \quad (7)$$

$$P(B \cap C_2 \cap \dots \cap C_n) = P(B) \prod_{j=2}^n P(C_j), \quad (8)$$

and $A = (A \setminus B) + B$, so we have

$$\begin{aligned}
 P(A \cap C_2 \cap \dots \cap C_n) &= P(A) \prod_{j=2}^n P(C_j) \\
 &= P((A \setminus B + B) \cap C_2 \cap \dots \cap C_n) \\
 &= P((A \setminus B) \cap C_2 \cap \dots \cap C_n) + P((B \cap C_2 \cap \dots \cap C_n)) \\
 &= P((A \setminus B) \cap C_2 \cap \dots \cap C_n) + P(B) \prod_{j=2}^n P(C_j).
 \end{aligned}$$

Hence, using (7) on the left side,

$$\begin{aligned}
 P((A \setminus B) \cap C_2 \cap \dots \cap C_n) &= P(A) \prod_{j=2}^n P(C_j) - P(B) \prod_{j=2}^n P(C_j) \\
 &= (P(A) - P(B)) \prod_{j=2}^n P(C_j) \\
 &= P(A \setminus B) \prod_{j=2}^n P(C_j).
 \end{aligned}$$

Thus $A \setminus B \in \mathcal{D}$.

(c) Suppose that $A_1, A_2, \dots \in \mathcal{D}$, and $A_n \nearrow A$. Then, since A_k is \nearrow

implies that the collection $\{A_k \setminus A_{k-1}\}$ is disjoint,

$$\begin{aligned}
& P(A \cap C_2 \cap \dots \cap C_n) \\
&= P(\cup_{k=1}^{\infty} A_k \cap C_2 \cap \dots \cap C_n) \\
&= P(\cup_{k=1}^{\infty} (A_k \setminus A_{k-1}) \cap C_2 \cap \dots \cap C_n) \\
&= \sum_{k=1}^{\infty} P((A_k \setminus A_{k-1}) \cap C_2 \cap \dots \cap C_n) \\
&= \sum_{k=1}^{\infty} P(A_k \setminus A_{k-1}) P(C_2) \cdots P(C_n) \\
&\quad \text{since } A_k \supset A_{k-1}, A_k, A_{k-1} \in \mathcal{D} \text{ implies } A_k \setminus A_{k-1} \in \mathcal{D} \text{ by (b)} \\
&= \left(\sum_{k=1}^{\infty} P(A_k \setminus A_{k-1}) \right) P(C_2) \cdots P(C_n) \\
&= P\left(\sum_{k=1}^{\infty} (A_k \setminus A_{k-1}) \right) P(C_2) \cdots P(C_n) \\
&= P(A) P(C_2) \cdots P(C_n).
\end{aligned}$$

Thus $A \in \mathcal{D}$, and we conclude that \mathcal{D} is a λ -system. Thus by the $\pi - \lambda$ theorem we conclude that $\sigma[\mathcal{C}_1] \subset \mathcal{D}$. Since the sets $C_i \in \mathcal{C}_i$ for $i = 2, \dots, n$ were arbitrary, this means that $\sigma[\mathcal{C}_1], \mathcal{C}_2, \dots, \mathcal{C}_n$ are independent. Next, fix $C_1 \in \sigma[\mathcal{C}_1], C_3 \in \mathcal{C}_3, \dots, C_n \in \mathcal{C}_n$ and repeat the above argument to show that

$$\sigma[\mathcal{C}_1], \sigma[\mathcal{C}_2], \mathcal{C}_3, \dots, \mathcal{C}_n$$

are independent. Repeating it $n - 2$ more times yields the conclusion.

4. Give an example of two collections of sets \mathcal{A}_1 and \mathcal{A}_2 that are independent but the generated σ -fields are not independent.

Solution: One example of this goes as follows: let $\Omega = \{1, 2, 3, 4\}$, and let $\mathcal{A} = 2^\Omega$. Suppose that $P(\{\omega\}) = 1/4$ for each $\omega \in \Omega$. Suppose that $\mathcal{A}_1 = \{\{1, 2\}\}$ and $\mathcal{A}_2 = \{\{2, 3\}, \{2, 4\}\}$. For $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$, we have

$$P(A_1 \cap A_2) = P(\{2\}) = 1/4 = (1/2)(1/2) = P(A_1)P(A_2)$$

so that $\mathcal{A}_1, \mathcal{A}_2$ are independent classes. Then we have $\{2\} \in \sigma[\mathcal{A}_2]$ (since $\{2\} = \{2, 3\} \cap \{2, 4\}$) and $\{1, 2\} \in \sigma[\mathcal{A}_1]$, but

$$P(\{1, 2\} \cap \{2\}) = P(\{2\}) = 1/4 \neq 1/8 = P(\{1, 2\})P(\{2\}).$$

Thus $\sigma[\mathcal{A}_1]$ and $\sigma[\mathcal{A}_2]$ are *not* independent classes. The difficulty here is that the class \mathcal{A}_2 is not a $\bar{\pi}$ -system.