

Statistics 521, Problem Set 7, Solutions

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1. Let μ be a sigma-finite measure and let ν be a finite measure on (Ω, \mathcal{A}) . Set $\phi = \mu - \nu$; i.e. define $\phi : \mathcal{A} \rightarrow (-\infty, \infty]$ by $\phi(A) = \mu(A) - \nu(A)$.
 - (a) Show that ϕ is a signed measure.
 - (b) Show that

$$\phi(A) = \int_A (f - g)d(\mu + \nu)$$

for some measurable functions f and g , $g \in \mathcal{L}_1(\mu + \nu)$. Thus ϕ can be written in the canonical form of the signed measure discussed in example 4.1.1 and the preceding problem.

- (c) Apply the results of Problem Set 6, problem # 5 to ϕ : compute ϕ^+ , ϕ^- , $|\phi|$, and $|\phi|(\Omega)$, assuming for the latter that μ is also a finite measure.

Solution: (a) First $\phi(\emptyset) = \mu(\emptyset) - \nu(\emptyset) = 0 - 0 = 0$; next, for $A \in \mathcal{A}$ we have $\phi(A) = \mu(A) - \nu(A) \in (-\infty, \infty]$ since $\nu(\Omega) < \infty$ and $\mu(A), \nu(A) \geq 0$; finally, for any collection of disjoint sets $A_n \in \mathcal{A}$,

$$\begin{aligned} \phi\left(\sum A_n\right) &= \mu\left(\sum A_n\right) - \nu\left(\sum A_n\right) = \sum \mu(A_n) - \sum \nu(A_n) \\ &= \sum (\mu(A_n) - \nu(A_n)) = \sum \phi(A_n). \end{aligned}$$

Thus ϕ is a signed measure.

- (b) Now $\mu \ll \mu + \nu$ and $\nu \ll \mu + \nu$, so by the Radon-Nikodym theorem, $f = \frac{d\mu}{d(\mu + \nu)}$ and $g = \frac{d\nu}{d(\mu + \nu)}$ exist, are measurable, and, for $A \in \mathcal{A}$,

$$\mu(A) = \int_A f d(\mu + \nu), \quad \nu(A) = \int_A g d(\mu + \nu).$$

Hence we have

$$\phi(A) = \mu(A) - \nu(A) = \int_A (f - g)d(\mu + \nu).$$

- (c) Note that in part (b) we have written ϕ in the form of the signed

measure treated in problem #1 with X replaced by $(f - g)$ and μ replaced by $\mu + \nu$. Hence it follows that

$$\phi^+(A) = \int_A (f - g)^+ d(\mu + \nu), \quad \phi^-(A) = \int_A (f - g)^- d(\mu + \nu)$$

and

$$|\phi|(A) = \int_A |f - g| d(\mu + \nu).$$

Thus we have $|\phi|(\Omega) = \int_\Omega |f - g| d(\mu + \nu)$.

2. For probability measures P and Q on (Ω, \mathcal{A}) , define

$$d_{TV}(P, Q) = \sup_{A \in \mathcal{A}} |P(A) - Q(A)|.$$

You showed in problem 2, problem set #6 that $d_{TV}(P, Q) = (1/2) \int |p - q| d\mu$ for any measure μ dominating both P and Q ; i.e. $P \ll \mu$, $Q \ll \mu$.

- (a) Show that $d_{TV}(P, Q)$ does not depend on the choice of μ .
- (b) Use the results of problem 1, part (c) to show that $d_{TV}(P, Q) = (1/2)|P - Q|(\Omega)$.

Solution: (a) This is actually pretty easy given the equality $d_{TV}(P, Q) \equiv \sup_{A \in \mathcal{A}} |P(A) - Q(A)| = (1/2) \int |p_\mu - q_\mu| d\mu$ for any measure μ dominating both P and Q : if μ and ν are two such measures then we have

$$\frac{1}{2} \int |p_\mu - q_\mu| d\mu = \sup_{A \in \mathcal{A}} |P(A) - Q(A)| = \frac{1}{2} \int |p_\nu - q_\nu| d\nu.$$

On the other hand, another proof which does not use this identity goes as follows. Since $\mu \ll \mu + \nu$ and $\nu \ll \mu + \nu$, we know that the Radon-Nikodym derivatives $f = d\mu/d(\mu + \nu)$ and $g = d\nu/d(\mu + \nu)$ exist. Furthermore, $P \ll \mu + \nu$ and $Q \ll \mu + \nu$. Hence

$$\begin{aligned} p_{\mu+\nu} &= \frac{dP}{d(\mu + \nu)} = \frac{dP}{d\mu} \frac{d\mu}{d(\mu + \nu)} = p_\mu f \\ &= \frac{dP}{d(\mu + \nu)} = \frac{dP}{d\nu} \frac{d\nu}{d(\mu + \nu)} = p_\nu g; \end{aligned}$$

similarly, $q_{\mu+\nu} = q_\mu f = q_\nu g$. Hence it follows that

$$\begin{aligned} \int |p_\mu - q_\mu| d\mu &= \int |p_\mu - q_\mu| f d(\mu + \nu) \\ &= \int |p_\mu f - q_\mu f| d(\mu + \nu) \\ &= \int |p_{\mu+\nu} - q_{\mu+\nu}| d(\mu + \nu) \\ &= \int |p_\nu - q_\nu| d\nu \end{aligned}$$

where the last inequality follows from the previous chain of equalities with μ replaced by ν and f replaced by g .

(b) From problem #2, part C, for any measure μ dominating both P and Q (e.g. $\mu = P+Q$) we have $|P-Q|(\Omega) = \int |p-q| d\mu = 2d_{TV}(P, Q)$.

3. Suppose that ϕ is a sigma-finite signed measure and $X \in L_1(|\phi|)$. The integral $\int X d\phi$ is defined by

$$\int X d\phi = \int X d\phi^+ - \int X d\phi^-.$$

Show that $|\int X d\phi| \leq \int |X| d|\phi|$.

Solution: Note that

$$\begin{aligned} \int_\Omega X d\phi &= \int X d\phi^+ - \int X d\phi^- \\ &= \int X^+ d\phi^+ - \int X^- d\phi^+ - \left(\int X^+ d\phi^- - \int X^- d\phi^- \right) \end{aligned}$$

where all integrals, $\int X^+ d\phi^+$, $\int X^- d\phi^+$, $\int X^+ d\phi^-$, and $\int X^- d\phi^-$ are ≥ 0 . Thus it follows that

$$\begin{aligned} \left| \int_\Omega X d\phi \right| &\leq \int X^+ d\phi^+ + \int X^- d\phi^+ \\ &\quad + \int X^+ d\phi^- + \int X^- d\phi^- \\ &= \int |X| d\phi^+ + \int |X| d\phi^- \\ &= \int |X| d|\phi|. \end{aligned}$$

4. PfS, Exercise 4.2.3, page 73: Flip a coin. If heads results, let X be a $\text{Uniform}(0, 1)$ random variable; if tails results, let X be a $\text{Poisson}(\lambda)$ random variable. The resulting distribution of X on \mathbb{R} is labeled ϕ .
- (a) Let μ denote Lebesgue measure on \mathbb{R} . Find the Lebesgue decomposition of ϕ with respect to μ ; that is, write $\phi = \phi_{ac} + \phi_s$.
- (b) Let μ be counting measure on $\{0, 1, 2, \dots\}$. Find the Lebesgue decomposition of ϕ with respect to μ .

Solution: First let $Y \sim \text{Bernoulli}(1/2)$ denote the coin toss variable with $P(Y = 1) = 1/2 = P(Y = 0)$. Then

$$P([X \leq x] \cap [Y = 1]) = (1/2)\{(x \vee 0) \wedge 1\},$$

$$P([X \leq x] \cap [Y = 0]) = (1/2) \sum_{k=0}^{[x]} \exp(-\lambda) \frac{\lambda^k}{k!},$$

for $x \in \mathbb{R}$, and ϕ is the measure on \mathbb{R} corresponding to the distribution function given by

$$F(x) = P(X \leq x) = (1/2)\{(x \vee 0) \wedge 1\} + (1/2) \sum_{k=0}^{[x]} \exp(-\lambda) \frac{\lambda^k}{k!}.$$

- (a) If μ is Lebesgue measure on \mathbb{R} , then the Poisson part of ϕ is singular with respect to μ , and the $\text{Uniform}(0, 1)$ part is absolutely continuous with respect to μ . Thus with

$$\phi_{ac}(A) = (1/2)\mu(A \cap [0, 1]) = \int_A \frac{1}{2} 1_{(0,1)}(z) d\mu(z) = \int_A \frac{1}{2} 1_{(0,1)}(z) dz,$$

$$\phi_s(A) \equiv \frac{1}{2} \sum_{k \in A} \exp(-\lambda) \frac{\lambda^k}{k!},$$

we have $\phi(A) = \phi_{ac}(A) + \phi_s(A)$ for all Borel sets A . Note that with $D \equiv \{0, 1, 2, \dots\}$ we have $\phi_{ac}(D) = 0$ and $\phi_s(D^c) = 0$.

- (b) If μ is counting measure on $D = \{0, 1, 2, \dots\}$, then the $\text{Uniform}(0, 1)$ part of ϕ is singular with respect to μ , and the Poisson part of ϕ is absolutely continuous with respect to μ . Thus with

$$\phi_s(A) = (1/2)\mu(A \cap [0, 1]) = \int_A \frac{1}{2} 1_{(0,1)}(z) dz,$$

$$\phi_{ac}(A) \equiv \frac{1}{2} \sum_{k \in A} \exp(-\lambda) \frac{\lambda^k}{k!} = \int_A \frac{1}{2} e^{-\lambda} \frac{\lambda^k}{k!} d\mu(k),$$

we can write $\phi(A) = \phi_{ac}(A) + \phi_s(A)$. Note that now $\phi_{ac}(D^c) = 0$ while $\phi_s(D) = 0$.

5. PfS, Exercise 4.4.3, page 84: Let F be \nearrow , right-continuous and bounded on \mathbb{R} with $F(-\infty) = 0$. Define μ_F via $\mu_F((a, b]) = F(b) - F(a)$ for all $a < b$. Show that $\mu_F \ll \lambda$ if and only if F is an absolutely continuous function on \mathbb{R} .

Solution: First suppose $\mu_F \ll \lambda$. Then by Theorem 4.2.1, page 72 (the Radon-Nikodym theorem), for all Borel sets A we have

$$\mu_F(A) = \int_A Z_0 d\lambda.$$

Since F is bounded and $F(-\infty) = 0$

$$\mu_F(\mathbb{R}) = \int_{\mathbb{R}} Z_0 d\lambda < \infty, \quad \text{and} \quad Z_0 \geq 0.$$

Thus $Z_0 \in \mathcal{L}_1(\lambda)$. By Theorem 3.2.5 (absolute continuity of the integral), for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that $\lambda(A) < \delta_\epsilon$ implies $\mu_F(A) = \int_A Z_0 d\lambda < \epsilon$. Take $A = \sum_{k=1}^n (c_k, d_k]$ with $\lambda(A) = \sum_{k=1}^n (d_k - c_k) < \delta_\epsilon$. Then

$$\epsilon > \int_A Z_0 d\lambda = \mu_F(A) = \mu_F\left(\sum_{k=1}^n (c_k, d_k]\right) = \sum_{k=1}^n |F(d_k) - F(c_k)|;$$

i.e. F is an absolutely continuous function on \mathbb{R} .

Conversely, suppose F is an absolutely continuous function on \mathbb{R} with $F(-\infty) = 0$, $F \nearrow$, and $F(\infty) < \infty$. Then, by Theorem 4.4.1 (the fundamental theorem of calculus), F' exists a.e. λ and

$$F(x) = F(x) - F(-\infty) = \int_{-\infty}^x F' d\lambda \quad \text{for all } x \in \mathbb{R}.$$

Thus if μ_F is the corresponding measure with $\mu_F((a, b]) = F(b) - F(a)$, it follows that for any Borel set A

$$\mu_F(A) = \int_A F' d\lambda.$$

But then $\mu_F \ll \lambda$ by the Radon-Nikodym theorem 4.2.1.