Statistics 521, Problem Set 6 Solutions

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1. PfS, Exercise 3.5.6, page 54, modified as follows: Suppose that $f_0, f_1, \ldots$ are ≥ 0, defined on a sigma-finite measure space $(\Omega, \mathcal{A}, \mu)$. (a) Suppose that $\int_\Omega f_n d\mu = 1$ for $n = 0, 1, \ldots$, and $f_n \rightarrow_{a.e.} f_0$ with respect to μ, then
$$\sup_{A \in \mathcal{A}} \Big| \int_A f_n d\mu - \int_A f_0 d\mu \Big| \rightarrow 0 \qquad \text{as} \ \ n \rightarrow \infty\,.$$
(b) Show that the conclusion of (a) holds if just $f_n \rightarrow_\mu f_0$ and $\int_\Omega f_n d\mu \rightarrow \int_\Omega f_0 d\mu$.

Solution: (a) By the solution to problem #2 below,
$$\sup_{A \in \mathcal{A}} \Big| \int_A f_n d\mu - \int_A f_0 d\mu \Big| = \int (f_0 - f_n)^+ d\mu$$
where $(f_0 - f_n)^+ \rightarrow_{a.e.} 0$ and is dominated by the integrable function f_0. Hence the right side converges to 0 by the dominated convergence theorem.

(b) If we have $f_n \rightarrow_\mu f_0$ and $\int f_n d\mu \rightarrow \int f_0 d\mu$, then we still have
$$\begin{aligned} & \sup_{A \in \mathcal{A}} \Big| \int_A f_n d\mu - \int_A f_0 d\mu \Big| && (1) \\ \leq \ & \sup_A \int_A |f_n - f_0| d\mu \\ \leq \ & \int_\Omega |f_n - f_0| d\mu = \int_\Omega |f_0 - f_n| d\mu \\ = \ & \int (f_0 - f_n)^+ d\mu + \int (f_0 - f_n)^- d\mu \\ = \ & \int (f_0 - f_n)^+ d\mu + \int (f_0 - f_n)^+ d\mu - D_n \\ = \ & 2 \int (f_0 - f_n)^+ d\mu - D_n && (2) \end{aligned}$$
where
$$D_n \equiv \int_\Omega (f_0 - f_n) d\mu = \int_\Omega \{(f_0 - f_n)^+ - (f_0 - f_n)^-\} d\mu \rightarrow 0\,.$$

But the right side of (2) converges to 0 by the dominated convergence theorem together with $D_n \to 0$.

2. Suppose that P, Q are two probability measures on the same measurable space $(\Omega, \mathcal{A})$ which are both absolutely continuous with respect to the measure μ with densities (Radon-Nikodym derivatives) p and q respectively. Thus $P(A) = \int_A p d\mu$ and $Q(A) = \int_A q d\mu$ for $A \in \mathcal{A}$. Show that

$$d_{TV}(P, Q) \equiv \sup_{A \in \mathcal{A}} |P(A) - Q(A)| = \frac{1}{2} \int |p - q| d\mu = \int (p - q)^+ d\mu \, .$$

   Solution: Let $\delta = p - q$, so that $\int_\Omega \delta d\mu = 0$. Then for $A \in \mathcal{A}$ we have $0 = \int_\Omega \delta d\mu = \int_A \delta d\mu + \int_{A^c} \delta d\mu$ and hence $| \int_{A^c} \delta d\mu| = | \int_A \delta d\mu|$. Thus for $A \in \mathcal{A}$ we have

$$2| \int_A \delta d\mu| = | \int_A \delta d\mu| + | \int_{A^c} \delta d\mu| \le \int_\Omega |\delta| d\mu \, .$$

   If $A = [\delta \ge 0]$, then we have equality in the above inequality, and hence it follows that

$$\sup_{A \in \mathcal{A}} |P(A) - Q(A)| = \sup_{A \in \mathcal{A}} | \int_A (p - q) d\mu| = \frac{1}{2} \int_\Omega |p - q| d\mu = \int (p - q)^+ d\mu \, .$$

3. Suppose that $X_n \sim$Binomial(n, p_n) for $n = 1, 2, \ldots$ with $np_n \to \lambda > 0$, and let P_n be the induced distribution of X_n on R. Let $X_0 \sim$Poisson(λ) and let P_0 be the corresponding induced distribution on R. Show that $d_{TV}(P_n, P_0) \to 0$ as $n \to \infty$

   Solution: This is an immediate consequence of the convergence of the Binomial probabilities to Poisson and problems #1 and #2 above: since

$$\begin{aligned} f_n(k) = P(X_n = k) &= \binom{n}{k} p_n^k (1 - p_n)^{n-k} \\ &\to \exp(-\lambda)(\lambda^k)/k! = P(X_0 = k) \equiv f_0(k) \, , \end{aligned}$$

where f_n and f_0 are the densities of P_n and P_0 with respect to counting measure μ on $\{0, 1, 2, \ldots\}$, it follows from problem 1 that

$$d_{TV}(P_n, P_0) = \frac{1}{2}\sum_{k=0}^{\infty} |P(X_n = k) - P(X_0 = k)| \to 0$$

as $n \to \infty$. (*Question(s):* how fast does this convergence happen? What if the Bernoulli rv's being added up have different success probabilities? What if there is some dependence between the Bernoulli's being added? ... For answers to these and more, see *Poisson Approximation*, by Barbour, Holst, and Janson, (1992).)

4. If X is a non-negative random variable satisfying $\int_0^\infty \sqrt{P(X > t)}dt < \infty$, then we say that $X \in \mathcal{L}_{2,1}$.
(a) Show that if $X \in \mathcal{L}_{2,1}$, then $X \in \mathcal{L}_2$.
(b) Show that if $X \in \mathcal{L}_r$ for some $r > 2$, then $X \in \mathcal{L}_{2,1}$.

Solution: (a) Suppose first that Y is a bounded random varible; i.e. $P(Y \le M) = 1$ for some constant M. Since the claimed inequality is trivially true if $E(Y^2) = 0$, we may assume that $E(Y^2) > 0$. Then $E(Y^2) < \infty$ and $\int_0^\infty \sqrt{P(Y > t)}dt < \infty$. Moreover, since $t^2 P(Y > t) \le E(Y^2)$ by Markov's inequality,

$$\begin{aligned} E(Y^2) &= \int_0^\infty 2tP(Y > t)dt \\ &= \int_0^\infty 2t\sqrt{P(Y > t)}\sqrt{P(Y > t)}dt \\ &\le 2\int_0^\infty \{E(Y^2)\}^{1/2}\sqrt{P(Y > t)}dt \end{aligned}$$

and hence, dividing by $\{E(Y^2)\}^{1/2} > 0$, it follows that

$$\{E(Y^2)\}^{1/2} \le 2\int_0^\infty \sqrt{P(Y > t)}dt\,.$$

For a general X with $X \ge 0$, set $Y \equiv Y_M \equiv X1_{[X\le M]}$. Y is a bounded random variable, and the above inequality yields

$$\{E(X^2 1_{[X\le M]})\}^{1/2} \le 2\int_0^\infty \sqrt{P(X1_{[X\le M]} > t)}dt\,.$$

Letting $M \nearrow \infty$ and applying the Monotone Convergence Theorem on both sides yields

$$\{E(X^2)\}^{1/2} \leq 2 \int_0^\infty \sqrt{P(X > t)}dt\,;$$

i.e. $||X||_2 \leq 2||X||_{2,1}$.

(b) If $X \in L_r(P)$ with $r > 2$, Markov's inequality yields $P(X \geq t) \leq E(X^r)/t^r$ for all $t > 0$, and hence we have

$$\begin{aligned} \int_0^\infty \sqrt{P(X > t)}dt &\leq \int_0^u dt + \int_u^\infty \frac{\{E(X^r)\}^{1/2}}{t^{r/2}}dt \\ &= u + (E(X^r))^{1/2}\frac{u^{-r/2+1}}{r/2-1}\,, \end{aligned}$$

for all $u > 0$. This bound is minimized by choosing $u = \{[E(X^r)]^{1/2}\}^{2/r} = ||X||_r$, and then we have

$$\int_0^\infty \sqrt{P(X > t)}dt \leq \frac{r}{r-2}||X||_r\,.$$

5. Suppose that $\phi(A) = \int_A X d\mu$ for X measurable, $X^- \in \mathcal{L}_1$. Identify ϕ^+, ϕ^-, $|\phi|$, and $|\phi|(\Omega)$ in this case.

Solution: I claim that

$$\begin{aligned} \phi^+(A) &= \int_A X^+ d\mu = \phi(A\Omega^+) \quad \text{with} \quad \Omega^+ = \{\omega : X(\omega) \geq 0\}, \\ \phi^-(A) &= \int_A X^- d\mu = -\phi(A\Omega^-) \quad \text{with} \quad \Omega^- = \{\omega : X(\omega) < 0\}, \\ |\phi|(A) &= \int_A |X| d\mu, \qquad \text{and} \\ |\phi|(\Omega) &= \int |X| d\mu. \end{aligned}$$

To see this, note that Ω^+, Ω^- are, respectively, positivity, negativity sets for ϕ since

$$\begin{aligned} \phi(A) &= \int_A X d\mu \geq 0 \qquad \text{for all events} \quad A \subset \Omega^+, \\ \phi(A) &= \int_A X d\mu \leq 0 \qquad \text{for all events} \quad A \subset \Omega^-. \end{aligned}$$

Furthermore, if $\tilde{\Omega}^+$, $\tilde{\Omega}^-$ denote the decomposition guaranteed by the Jordan-Hahn theorem 1.1, then

$$\begin{aligned}
&\phi(\Omega^+ \setminus \tilde{\Omega}^+) = \phi(\Omega^+ \cap \tilde{\Omega}^-) = 0, \qquad \text{and} \\
&\phi(\tilde{\Omega}^+ \setminus \Omega^+) = \phi(\tilde{\Omega}^+ \cap \Omega^-) = 0,
\end{aligned}$$

where the zeroes follow by using the definitions of Ω^+, Ω^-, $\tilde{\Omega}^+$, $\tilde{\Omega}^-$. Thus

$$|\phi|(\Omega^+ \Delta \tilde{\Omega}^+) = 0;$$

i.e. $\Omega^+ = [X \geq 0]$ differs from $\tilde{\Omega}^+$ by (at most) a set of $|\phi|-$measure 0.