Statistics 521, Problem Set 4 Solutions

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1. PfS, Exercise 2.3.3, page 32: show that $X_n \to_\mu X$ if and only if $X_n - X_m \to_\mu 0$.

   Solution: First suppose that $X_n \to_\mu X$. Then for every $\epsilon > 0$ we have

   $$\mu([|X_m - X_n| > \epsilon]) \le \mu([|X_m - X| > \epsilon/2]) + \mu([|X_n - X| > \epsilon/2]) \to 0$$

   as $m, n \to \infty$; i.e. $X_n - X_m \to_\mu 0$.

   Now suppose that $X_n - X_m \to_\mu 0$. First, choose a subsequence n_k increasing so that

   $$\mu([|X_{n_k} - X_l| > 2^{-k}]) < 2^{-k} \qquad \text{for all } \ l > n_k, .$$

   Set $B_m \equiv \cup_{k=m}^{\infty} A_k$, and note that

   $$\mu(B_m) \le \sum_{k=m}^{\infty} \mu(A_k) < \sum_{m}^{\infty} 2^{-k} = 2^{-(m-1)} .$$

   On $B_m^c = \cap_m^\infty A_k^c$ we have $|X_{n_k} - X_{n_{k+1}}| \le 2^{-k}$ for all $k \ge m$. Moreover, for $n_i > n_j > m$ it follows that

   $$|X_{n_i}(\omega) - X_{n_j}(\omega)| \le \sum_{k=j}^{\infty} |X_{n_k}(\omega) - X_{n_{k+1}}(\omega)| < 2^{-(j-1)}$$

   for $\omega \in B_m^c$, and this implies that $X_{n_k}(\omega) \to X(\omega)$ for all $\omega \in C \equiv \cup_1^\infty B_m^c$ with

   $$\mu(C^c) = \mu(\cap_1^\infty B_m) \le \limsup \mu(B_m) \le \lim 2^{-(m-1)} = 0 .$$

   Define $X(\omega) = 0$ for $\omega \in C^c$; then X is measurable, and we have

   $$\mu([|X_n - X| \ge \epsilon) \le \mu([|X_n - X_{n_k}| \ge \epsilon/2) + \mu([|X_{n_k} - X| \ge \epsilon/2) \to 0$$

   as $n \ge n_k \to \infty$.

2. PfS, Exercise 2.3.4, page 32: (a) Suppose that $\mu(\Omega) < \infty$ and g is continuous a.e. μ_X. Then $X_n \to_\mu X$ implies $g(X_n) \to_\mu g(X)$.
(b) Let g be uniformly continuous on the real line. Then $X_n \to_\mu X$ implies that $g(X_n) \to_\mu g(X)$. (Here $\mu(\Omega) = \infty$ is allowed.)

Solution: (a) Let $\{n'\}$ be a subsequence. We want to show that for some subsequence $\{n''\}$ it follows that $g(X_{n''}) \to_{a.e.} g(X)$. Then by (15) of Theorem 2.3.1 it follows that $g(X_n) \to_\mu g(X)$. But since $X_n \to_\mu X$ we know, by (15) of Theorem 2.3.1, that there is a further subsequence $\{n''\}$ such that $X_{n''} \to_{a.e.} X$. For this subsequence we have $g(X_{n''}) \to_{a.e.} g(X)$ (by restricting in addition to the set $[X \in C_g]$ with $\mu([X \in C_g^c]) = \mu_X(C_g^c) = 0$ for which g is continuous). Thus we conclude that $g(X_n) \to_\mu g(X)$.
(b) Let $\epsilon > 0$. Since g is uniformly continuous there is a $\delta = \delta_\epsilon$ such that $|y - x| < \delta_\epsilon$ implies $|g(y) - g(x)| < \epsilon$. Since $X_n \to_\mu X$, for every $\gamma > 0$ there exists an $N = N_{\epsilon,\gamma}$ such that

$$\mu([|X_n - X| \geq \delta_\epsilon]) < \gamma, \qquad \text{for all } n \geq N_{\epsilon,\gamma}.$$

Then we have

$$\begin{aligned} &\mu([|g(X_n) - g(X)| \geq \epsilon]) \\ &\quad = \mu([|g(X_n) - g(X)| \geq \epsilon] \cap [|X_n - X| \geq \delta_\epsilon]) \\ &\qquad\qquad + \mu([|g(X_n) - g(X)| \geq \epsilon] \cap [|X_n - X| < \delta_\epsilon]) \\ &\quad \leq \mu([|X_n - X| \geq \delta_\epsilon]) + \mu(\emptyset) \\ &\quad \leq \gamma + 0 = \gamma \qquad \text{for } \ n \geq N_{\epsilon,\gamma}. \end{aligned}$$

Thus $\mu([|g(X_n) - g(X)| \geq \epsilon]) \to 0$ as $n \to \infty$; i.e. $g(X_n) \to_\mu g(X)$.

3. PfS, Exercise 3.2.1, page 42: If $X \geq 0$ and $\int X d\mu = 0$, then $\mu([X > 0]) = 0$.

Solution: Let $\epsilon > 0$. Then $X \geq \epsilon 1_{[X \geq \epsilon]}$, and hence

$$0 = \int X d\mu \geq \epsilon \mu([X \geq \epsilon])$$

Since $[X > 0] = \cup_{n=1}^\infty [X \geq 1/n] = \lim [X \geq 1/n]$, We find that $\mu([X > 0]) = \lim_n \mu([X \geq 1/n]) = \lim_n 0 = 0\,.$

4. PfS, Exercise 3.2.2, page 42: Show that

$$\int_A X d\mu \begin{cases} = 0 \\ \geq 0 \end{cases} \qquad \text{for all } A \in \mathcal{A}, \qquad \text{implies} \qquad X \begin{cases} = 0 \ a.e. \\ \geq 0 \ a.e. \end{cases}$$

Solution: Suppose first that $\int_A X d\mu = 0$ for all $A \in \mathcal{A}$. Then with $A = [X^+ \geq 0]$ we have $0 = \int X 1_{[X^+ \geq 0]} d\mu = \int Y d\mu$ where $Y \equiv X 1_{[X^+ \geq 0]} = X^+ \geq 0$. Then by the previous exercise, $0 = \mu([Y > 0]) = \mu([X^+ > 0])$; i.e $X^+ = 0$ a.e. Similarly, choosing $A = [X^- \geq 0]$ yields $X^- = 0$ a.e.; combining the two results gives $X = X^+ - X^- = 0$ a.e.
Now suppose that $\int_A X d\mu \geq 0$ for all $A \in \mathcal{A}$. Taking $A = [X < 0] = [X^- > 0]$ yields $0 \leq \int_A X d\mu = \int -X^- d\mu \leq 0$ since $X^- \geq 0$. Thus $\int X^- d\mu = 0$. By problem 3 this implies $\mu([X < 0]) = \mu([X^- > 0]) = 0$. Hence $X \geq 0$ a.e. μ.

5. The Cantor singular distribution function F is the function $F : [0,1] \to [0,1]$ defined as follows: $F(x) = 1/2$ for $x \in (1/3, 2/3)$; $F(x) = 1/4$ for $x \in (1/9, 2/9)$ and $F(x) = 3/4$ for $x \in (7/9, 8/9)$; $\ldots$; $F(x) = 1/2^n, 3/2^n, 5/2^n, \ldots$ on the successive intervals removed from K_{n-1} in the construction of K_n. Thus F is defined on the open set $[0,1] \setminus K$, is nondecreasing, and has values in $[0,1]$. Extend it to all of $[0,1]$ by letting $F(0) = 0$, and setting

$$F(x) \equiv \sup\{F(y) : y \in [0,1] \setminus K \ \text{ and } \ y < x\}$$

for $x \in K$ and $x \neq 0$. Show that F is non-decreasing and continuous with $F(0) = 0$ and $F(1) = 1$. Because F is continuous, its range is all of $[0,1]$. Now the inverse (or quantile) function F^{-1} of F defined by

$$F^{-1}(y) \equiv \inf\{x \in [0,1] : \ F(x)] \geq y\}$$

is one-to-one (injective) and $F^{-1}([0,1]) \subset K$. Show that F^{-1} is Borel-measurable.

Solution: First, F is nondecreasing: for $x < y$ with $x, y \in K^c$, it is clearly true that $F(x) \leq F(y)$ from the definition of F on K^c.
If $x \in K^c$, $y \in K$, then $F(y) = \sup\{F(t)) : t \in K^c, t < y\} \geq F(x)$.
If $x \in K$, $y \in K$, $x < y$, then

$$F(y) = \sup\{F(t)) : t \in K^c, t < y\} \geq \sup\{F(t)) : t \in K^c, t < x\} = F(x)$$

since the set over which the first supremum is taken contains the set over which the second supremum is taken.

Finally, if $x \in K$, $y \in K^c$, consider $t < y$, $t \in K^c$. By the first case, $F(t) \leq F(y)$. But by the same inclusion argument

$$F(x) = \sup\{F(t)) : t \in K^c, t < x\} \leq \sup\{F(t)) : t \in K^c, t < y\} \leq F(y)\,.$$

Hence F is nondecreasing on $[0,1]$.

To show that F is continuous on $[0,1]$: note that the Cantor set K can be expressed as

$$K = \{x \in [0,1] : x = \sum_{n=1}^{\infty} 2a_n 3^{-n} : \quad a_n \in \{0,1\}\}\,.$$

Moreover, the supremum definition of F implies that for $x \in K$,

$$F(\sum_{n=1}^{\infty} 2a_n 3^{-n}) = \sum_{n=1}^{\infty} a_n 2^{-n}\,.$$

Note that $\{F(x) : x \in K\} = [0,1]$. Since F is nondecreasing and achieves each value in $[0,1]$ it is continuous.

Finally, note that $F^{-1}(u)$ is monotone non-decreasing and left-continuous with jumps at the (countably many) flat intervals of F. Since $g \equiv F^{-1}$ is monotone, it follows that $g^{-1}(-\infty, x]$ is an interval of the form $[0,t)$ or $[0,t]$ which are in $\mathcal{B}$. It follows that $g = F^{-1}$ is Borel-measurable.