**Statistics 521, Problem Set 2 Solutions**

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1. PfS, Exercise 1.1.3, page 9: (a) The minimal $\lambda-$system generated by the class $\mathcal{D}$ is denoted by $\lambda[\mathcal{D}]$. Show that $\lambda[\mathcal{D}]$ is equal to the intersection of all $\lambda-$systems containing $\mathcal{D}$.
(b) A collection $\mathcal{A}$ of subsets of $\Omega$ is a $\sigma-$field if and only if it is both a $\pi-$system and a $\lambda-$system.
(c) Let $\mathcal{C}$ be a $\pi-$system and let $\mathcal{D}$ be a $\lambda-$system. Then $\mathcal{C} \subset \mathcal{D}$ implies that $\sigma[\mathcal{C}] \subset \mathcal{D}$.

**Solution:** (a) Let

$$\lambda[\mathcal{D}] \equiv \cap\{\mathcal{F}_\alpha : \ \mathcal{F}_\alpha \text{ is a } \lambda - \text{system with } \mathcal{D} \subset \mathcal{F}_\alpha\}.$$

Now $\Omega \in \mathcal{F}_\alpha$ for all $\alpha$, so $\Omega \in \lambda[\mathcal{D}]$. Further, if $A, B \in \lambda[\mathcal{D}]$ with $B \subset A$, then $B, A \in \mathcal{F}_\alpha$ for all $\alpha$ and hence $A \setminus B \in \mathcal{F}_\alpha$ for all $\alpha$, and hence $A \setminus B \in \lambda[\mathcal{D}]$. Finally, if $\{A_1, A_2, \ldots\}$ is an increasing family of sets in $\lambda[\mathcal{D}]$, then $\{A_1, A_2, \ldots\} \subset \mathcal{F}_\alpha$ for all $\alpha$. Hence $\lim_n A_n \in \mathcal{F}_\alpha$ for all $\alpha$, and hence $\lim_n A_n \in \mathcal{F}_\alpha$ for all $\alpha$. Thus it follows that $\lim_n A_n \in \lambda[\mathcal{D}]$. Thus $\lambda[\mathcal{D}]$ is a $\lambda-$system. If $\mathcal{A}'$ is a $\lambda-$system such that $\mathcal{D} \subset \mathcal{A}'$, then $\mathcal{A} = \mathcal{F}_\alpha$ for some $\alpha$, and hence $\lambda[\mathcal{D}] \subset \mathcal{A}'$; i.e. $\lambda[\mathcal{D}]$ is the minimal $\lambda-$system containing $\mathcal{D}$.
(b) Suppose first that $\mathcal{A}$ is a $\sigma-$field. Thus it is closed under countable intersections, and hence, in particular it is closed under finite intersections and is a $\pi-$system. To show that $\mathcal{A}$ is a $\lambda-$system, first consider $A, B \in \mathcal{A}$ with $A \subset B$. Since $\mathcal{A}$ is closed under complementation, $A^c \in \mathcal{A}$, and hence also $B \cap A^c = B \setminus A \in \mathcal{A}$. Also $\Omega \in \mathcal{A}$ since it is a $\sigma-$field. Finally, if $\{A_n\}$ is a sequence of sets in $\mathcal{A}$ with $A_n \uparrow A$, then $A = \lim_n A_n = \cup_{n=1}^\infty A_n \in \mathcal{A}$ since $\mathcal{A}$ is a $\sigma-$field. Thus $\mathcal{A}$ is a $\lambda-$system, and this completes the proof that a $\sigma-$field is both a $\pi-$system and a $\lambda-$system.

Now suppose that $\mathcal{A}$ is a $\pi-$system and a $\lambda-$system. Let $A \in \mathcal{A}$. Since $\mathcal{A}$ is a $\lambda-$system, $\Omega \in \mathcal{A}$. Since $A \subset \Omega$ and $\mathcal{A}$ is a $\lambda-$system, $A^c = \Omega \cap C^c = \Omega \setminus A \in \mathcal{A}$. To show that $\mathcal{A}$ is closed under countable

unions, suppose that $\{A_n\}$ is a countable family of sets with $A_n \in \mathcal{A}$ for each $n$. Set $B_n \equiv \cup_{i=1}^n A_i$. Then $B_n \in \mathcal{A}$ for each $n$ since $A, B \in \mathcal{A}$ implies that $A \cup B = (A^c \setminus B)^c \in \mathcal{A}$ since $\mathcal{A}$ is a $\pi-$ system and a $\lambda-$system implies that it is closed under intersections, complements, and set differences. But since $\mathcal{A}$ is a $\lambda-$system this implies that $\cup_{n=1}^\infty = \lim_n \cup_{i=1}^n A_i = \lim_n \cup_{i=1}^n B_i = \lim_n B_n \in \mathcal{A}$, and hence $\mathcal{A}$ is closed under countable unions. Hence $\mathcal{A}$ is a $\sigma-$field.
(c) It is clear that $\lambda(\mathcal{C}) \subset \sigma[\mathcal{C}]$ (since there are fewer restrictions in defining a $\lambda-$system than a field; or from (b)). If we show that $\lambda[\mathcal{C}]$ is a $\pi-$system, then since $\sigma[\mathcal{C}]$ is a $\lambda-$system containing $\mathcal{C}$, it must also be the minimal $\lambda-$system containing $\mathcal{C}$ and hence $\sigma[\mathcal{C}] = \lambda[\mathcal{C}] \subset \lambda[\mathcal{D}] \subset \mathcal{D}$. Thus it suffices to show that $\lambda[\mathcal{C}]$ is a $\pi-$system.

We do this in two steps:
**Step 1:** Let

$$\mathcal{D}_1 = \{B \in \lambda(\mathcal{C}) : \quad B \cap C \in \lambda(\mathcal{C}) \ \text{ for all } \ C \in \mathcal{C}\}$$

where $\mathcal{C}$ is a $\pi-$system and $\lambda(\mathcal{C})$ is the smallest $\lambda-$system containing $\mathcal{C}$. To show that $\mathcal{D}_1$ is a $\lambda-$system we need to show that:
(i) $\Omega \in \mathcal{D}_1$.
(ii) If $D_n \in \mathcal{D}_1$, $D_n \uparrow$, then $\cup D_n \in \mathcal{D}_1$.
(iii) If $A, B \in \mathcal{D}_1$ with $A \subset B$, then $A \setminus B \in \mathcal{D}_1$.
Proof of (i): $\Omega \in \lambda(\mathcal{C})$ since it is a $\lambda-$system, so we have $\Omega \cap C = C \in \mathcal{C} \subset \lambda(\mathcal{C})$ for each $C \in \mathcal{C}$, and hence $\Omega \in \mathcal{D}_1$.
Proof of (ii): Suppose $D_1, D_2, \ldots \in \mathcal{D}_1$ and $D_n \uparrow$. Then we have $\cup_n D_n \in \lambda(\mathcal{D})$ (since each $D_n \in \lambda(\mathcal{D})$, a $\lambda-$system), and $(\cup_n D_n) \cap C = \cup_n (D_n \cap C) = \cup_n B_n \in \lambda(\mathcal{C})$ since $B_n \in \lambda(\mathcal{C})$ is $\uparrow$. Hence $\cup_n D_n \in \mathcal{D}_1$.
Proof of (iii): Suppose $A, B \in \mathcal{D}_1$ with $B \subset A$. Then $AC, BC, A, B \in \lambda(\mathcal{C})$ for all $C \in \mathcal{C}$, so

$$(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C) \in \lambda(\mathcal{C})$$

for all $C \in \mathcal{C}$. Hence $A \setminus B \in \mathcal{D}_1$.

**Step 2:** Let

$$\mathcal{D}_2 \equiv \{A \in \lambda[\mathcal{C}] : B \cap A \in \lambda[\mathcal{C}], \text{ for all } B \in \lambda[\mathcal{C}]\}.$$

Step 1 showed that $\mathcal{D}_2$ contains $\mathcal{C}$. As in step 1, we can show that $\mathcal{D}_2$ inherits the $\lambda-$system structure form $\lambda[\mathcal{C}]$ and that therefore $\mathcal{D}_2 = \lambda[\mathcal{C}]$. But the fact that $\lambda[\mathcal{C}] = \mathcal{D}_2$ means that $\lambda[\mathcal{C}]$ is a $\pi-$system.

2. PfS, Exercise 1.2.1, page 15. Let $(\Omega, \mathcal{A}, \mu)$ denote a measure space. Show that

$$\begin{aligned} \widehat{\mathcal{A}}_\mu &\equiv \{A : A_1 \subset A \subset A_2,\ A_1, A_2 \in \mathcal{A}, \mu(A_2 \setminus A_1) = 0\} \\ &= \{A \cup N : A \in \mathcal{A} \text{ and } N \subset (\text{some } B) \in \mathcal{A} \ \text{ with } \ \mu(B) = 0\} \\ &= \{A \triangle N : \ \ A \in \mathcal{A}, \ \ N \subset (\text{some } B) \in \mathcal{A} \ \text{ with } \ \mu(B) = 0\} \end{aligned}$$

and is a $\sigma-$field. Show that $(\Omega, \widehat{\mathcal{A}}_\mu, \widehat{\mu})$ is complete.

**Solution:** Let these three collections of sets be called $\widehat{\mathcal{A}}_1$, $\widehat{\mathcal{A}}_2$, and $\widehat{\mathcal{A}}_3$ respectively.
To show that $\widehat{\mathcal{A}}_1 \subset \widehat{\mathcal{A}}_2$, suppose that $D \in \widehat{\mathcal{A}}_1$. Then there exist sets $A_1, A_2 \in \mathcal{A}$ such that $A_1 \subset D \subset A_2$ and $\mu(A_2 \setminus A_1) = 0$. Let $B = A_2 \setminus A_1$; then $\mu(B) = 0$ and $B \in \mathcal{A}$. Furthemore $D = A_1 \cup N$ for some $N \subset B$. Hence with $A = A_1$ and $B = A_2 \setminus A_1$ we have $D = A \cup N$ where $A \in \mathcal{A}$ and $N \subset B$ with $\mu(B) = 0$. Thus $D \in \widehat{\mathcal{A}}_2$, and $\widehat{\mathcal{A}}_1 \subset \widehat{\mathcal{A}}_2$.
To show that $\widehat{\mathcal{A}}_2 \subset \widehat{\mathcal{A}}_3$, let $D \in \widehat{\mathcal{A}}_2$. Then $D = A \cup N$ with $A \in \mathcal{A}$, $N \subset B \in \mathcal{A}$ having $\mu(B) = 0$, so $A \subset D$. Let $N_1 \equiv D \setminus A = D \cap A^c = N \cap A^c \subset N \subset B$. Hence we have

$$\begin{aligned} A \triangle N_1 &= (A^c \cap (D \cap A^c)) \cup (A \cap (D \cap A^c)^c) \\ &= (D \cap A^c) \cup ((A \cap D^c) \cup A) \\ &= (D \cap A^c) \cup A = D \cup A = D \end{aligned}$$

and hence $D \in \widehat{\mathcal{A}}_3$.
To show that $\widehat{\mathcal{A}}_3 \subset \widehat{\mathcal{A}}_1$, let $D \in \widehat{\mathcal{A}}_3$. Then $D = A \triangle N$ with $A \in \mathcal{A}$ and $N \subset B \in \mathcal{A}$ with $\mu(B) = 0$. Take $A_1 = A \cap B^c$, $A_2 = A \cup B$. Then $A_1 \subset A \cap N^c \subset D \subset A \cup N \subset A_2$. Since $A_2 \setminus A_2 = (A \cup B) \cap (A \cap B^c)^c = A \cap (A^c \cup B) \cup B \cap A^c \cap B = (A \cap B) \cup (A^c \cap B) = B$ so that $\mu(A_2 \setminus A_1) = \mu(B) = 0$, it follows that $D \in \widehat{\mathcal{A}}_1$, and hence that $\widehat{\mathcal{A}}_3 \subset \widehat{\mathcal{A}}_1$.

Since we have shown that

$$\widehat{\mathcal{A}}_1 \subset \widehat{\mathcal{A}}_2 \subset \widehat{\mathcal{A}}_3 \subset \widehat{\mathcal{A}}_1 ,$$

it follows that $\widehat{\mathcal{A}}_1 = \widehat{\mathcal{A}}_2 = \widehat{\mathcal{A}}_3$.

To show that $\widehat{\mathcal{A}}_\mu$ is a $\sigma-$field:
Let $A \in \widehat{\mathcal{A}}_1$. Then $A_1 \subset A \subset A_2$, so that $A_2^c \subset A^c \subset A_1^c$ with $A_2^c, A_1^c \in$

$\mathcal{A}$, and where $A_1^c \setminus A_2^c = A_2 \setminus A_1$ and hence $\mu(A_1^c \setminus A_2^c) = \mu(A_2 \setminus A_1) = 0$. Hence $A^c \in \widehat{\mathcal{A}}_1 = \widehat{\mathcal{A}}_\mu$.
Let $D_1, \ldots, D_n \in \widehat{\mathcal{A}}_2$. Then $D_n = A_n \cup N_n$ with $A_n \in \mathcal{A}$, $N_n \subset B_n \in \mathcal{A}$ with $\mu(B_n) = 0$ for each $n$. Thus $\cup A_n \in \mathcal{A}$, $\cup B_n \in \mathcal{A}$, $\cup N_n \subset \cup B_n$, and $\mu(\cup B_n) \leq \sum \mu(B_n) = 0$. Hence $\cup D_n = (\cup A_n) \cup (\cup N_n) \in \widehat{\mathcal{A}}_2$. Hence $\widehat{\mathcal{A}}_\mu$ is a $\sigma-$field.

To show that $(\Omega, \widehat{\mathcal{A}}_\mu, \widehat{\mu})$ is complete, first let $A_1 \cup N_1 = A_2 \cup N_2$ with $A_1, A_2 \in \mathcal{A}$, $N_1 \subset B_1$, $N_2 \subset B_2$ with $\mu(B_1) = \mu(B_2) = 0$. By definition $\widehat{\mu}(A_1 \cup N_1) = \mu(A_1)$ $\widehat{\mu}(A_2 \cup N_2) = \mu(A_2)$. But $A_1 \subset A_1 \cup N_1 = A_2 \cup N_2 \subset A_2 \cup B_2$ and similarly $A_2 \subset A_1 \cup B_1$. Hence we have $\mu(A_1) \leq \mu(A_2) + \mu(B_2) = \mu(A_2)$ and $\mu(A_2) \leq \mu(A_1) + \mu(B_1) = \mu(A_1)$, or $\mu(A_1) = \mu(A_2)$. Thus $\widehat{\mu}$ is well-defined. That it extends $\mu$ is trivial. To show completeness, let $D \subset (\text{some } \; B) \in \widehat{\mathcal{A}}_\mu$ with $\mu(B) = 0$. Then $D = \emptyset \cup D \in \widehat{\mathcal{A}}_2 = \widehat{\mathcal{A}}_\mu$.

3. PfS, Exercise 1.2.3, page 16: Suppose that $\mu$ on a field $\mathcal{C}$ is $\sigma-$finite on $\mathcal{C}$ and is extended to $\mathcal{A} = \sigma(\mathcal{C})$; call the extension $\mu$.
(a) For each $A \in \mathcal{A}$ with $\mu(A) < \infty$ and each $\epsilon > 0$ there exists a set $C = C_\epsilon \in \mathcal{C}$ such that $\mu(A \triangle C) < \epsilon$.
(b) Let $\mu$ denote counting measure on the integers. Then $\mathcal{C} = \{C : C \text{ or } C^c \text{ is finite}\}$ is a field. Determine $\sigma[\mathcal{C}]$. Show that the conclusion of part (a) fails for the set of even integers.

Proof of (a): Now

$$\mu(A) = \inf\{\sum_n \mu(A_n) : A \subset \cup_1^\infty A_n \;\; \text{with all} \;\; A_n \in \mathcal{C}\}.$$

Hence there exists $\{A_n\} \subset \mathcal{C}$ such that

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu(A) + \epsilon/2\,.$$

Without loss of generality, we may assume that the sets $A_n$ are disjoint (if not, form the disjoint sets $B_1 = A_1$, $B_n = A_1^c \cap \ldots \cap A_{n-1}^c \cap A_n$, $n = 2, 3, \ldots$). Furthermore, there exists an $N = N_\epsilon$ sufficiently large such that $\sum_{N+1}^{\infty} \mu(A_n) \leq \epsilon/2$, and hence

$$\sum_{n=1}^{\infty} \mu(A_n) < \sum_{n=1}^{N} \mu(A_n) + \epsilon/2 = \mu(\sum_{n=1}^{N} A_n) + \epsilon/2\,.$$

Then $C \equiv \sum_{n=1}^{N} A_n \in \mathcal{C}$,

$$\mu(A \setminus C) \leq \mu(\sum_{n=1}^{\infty} A_n \setminus C) = \mu(\sum_{n=1}^{\infty} A_n) - \mu(C) < \epsilon/2$$

by the choice of $N$, and

$$\mu(C \setminus A) \leq \mu(\sum_{n=1}^{\infty} A_n \setminus A) = \mu(\sum_{n=1}^{\infty} A_n) - \mu(A) < \epsilon/2$$

by the choice of $\{A_n\}$. Putting these together gives

$$\mu(A \bigtriangleup C) = \mu(A \setminus C) + \mu(C \setminus A) < \epsilon/2 + \epsilon/2 = \epsilon.$$

(b) Note that all the singletons $D_k = \{k\}$ are in $\mathcal{C}$, and since all subsets of $\mathbb{Z}$ are either finite or countable, every subset $A$ of $\mathbb{Z}$ can be written as a countable union of the singletons $D_k$, $k \in A$. Thus $\sigma[\mathcal{C}] = 2^{\mathbb{Z}}$. Consider $A = \{2, 4, 6, \ldots\} = \cup_k \{2k\} \equiv \cup_k C_k$ where each $C_k \in \mathcal{C}$ since $C_k$ itself is a finite set. Note that $A \notin \mathcal{C}$ since neither $A$ nor $A^c = \{1, 3, \ldots\}$ is finite. Thus $\mu(A) = \infty$ (so the hypothesis of (a) fails). Furthermore, for any set $C \in \mathcal{C}$

$$A \Delta C = (A \cap C^c) \cup (A^c \cap C)$$

where both $A$ and $A^c$ are non-finite sets, and either $C$ or $C^c$ is non-finite, and hence at least one of $A \cap C^c$ and $A^c \cap C$ is also non-finite. Thus $\mu(A \Delta C) = \infty$ for all $C \in \mathcal{C}$. Hence the conclusion of (a) fails to hold.

4. PfS, Exercise 1.2.4, page 16. Let $\Omega$ consist of the sixteen values $1, \ldots, 16$. (Think of them arranged in four rows of four values.) Let

$$\begin{aligned}
C_1 &= \{1, 2, 3, 4, 5, 6, 7, 8\},\\
C_2 &= \{9, 10, 11, 12, 13, 14, 15, 16\},\\
C_3 &= \{1, 2, 5, 6, 9, 10, 13, 14\},\\
C_4 &= \{3, 4, 7, 8, 11, 12, 15, 16\}.
\end{aligned}$$

Let $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$, and let $\mathcal{A} = \sigma[\mathcal{C}]$.

(a) Show that $\mathcal{A} \equiv \sigma[\mathcal{C}] \neq 2^\Omega$.
*Proof.* Write $\{1, \ldots, 16\}$ in four rows of four numbers each as follows:

| | $C_3$ | | $C_4$ | |
|---|---|---|---|---|
| $C_1$ | 1 | 2 | 3 | 4 |
| | 5 | 6 | 7 | 8 |
| $C_2$ | 9 | 10 | 11 | 12 |
| | 13 | 14 | 15 | 16 |

Let $\mathcal{B} \equiv \{C_i \cap C_j : \ i, j \in \{1, \ldots, 4\}\}$. Then it is clear that $\#(\sigma[\mathcal{C}]) = 2^{\mathcal{B}} = 2^4 \neq 2^{16} = \#(2^\Omega) = 2^{16}$. Thus $\sigma[\mathcal{C}] \neq 2^\Omega$.

(b) Let $\mu(C_i) = 1/2$ where $1 \le i \le 4$ with $\mu(C_1C_3) = 1/4$. Show that $\hat{\mathcal{A}}_\mu = \mathcal{A}$ with $2^4$ sets.
*Proof.* Let $B_1 = C_1C_3$, $B_2 = C_1C_4$, $B_3 = C_2C_3$, and $B_4 = C_2C_4$. Then $\mu(B_1) = \mu(C_1C_3) = 1/4$ implies that $\mu(B_i) = 1/4$ for $i = 2, 3, 4$. This holds since $1/2 = \mu(C_1) = \mu(B_1 + B_2) = \mu(B_1) + \mu(B_2) = 1/4 + \mu(B_2)$, so that $\mu(B_2) = 1/4$, and similarly $1/2 = \mu(C_3) = \mu(B_1 + B_3) = \mu(B_1) + \mu(B_3) = 1/4 + \mu(B_3)$, so $\mu(B_3) = 1/4$, and $1/2 = \mu(C_4) = \mu(B_2 + B_4) = \mu(B_2) + \mu(B_4) = 1/4 + \mu(B_4)$, so $\mu(B_4) = 1/4$. Thus the only set $B \in \mathcal{A}$ with $\mu(B) = 0$ is $B = \emptyset$, and it follows that $\hat{\mathcal{A}}_\mu = \mathcal{A}$ with $\#(\hat{\mathcal{A}}_\mu) = \#(\mathcal{A}) = 2^4$.

(c) Now suppose $\mu(C_i) = 1/2$ for $i = 2, 3, 4$, but $\mu(C_2C_4) = 0$. Show that $\hat{\mathcal{A}}_\mu$ contains $2^{10} = 1024$ sets.
*Proof.* In this case $\mu(B_4) = \mu(C_2C_4) = 0$, and this implies that $\mu(B_2) = 1/2 = \mu(B_3)$ (since $\mu(C_2) = 1/2 = \mu(C_4)$. Thus we also have $\mu(B_1) = 0$ (since $\mu(B_1) + \mu(B_2) = 1/2$). Therefore we need to consider all the sets $N \subset 2^{B_1} + 2^{B_4}$ in forming the completion; that is we need to consider all the sets $\{\{1\}, \{2\}, \{5\}, \{6\}, \{11\}, \{12\}, \{14\}, \{16\}\}$ in forming the completion together with the two basic sets with non-zero probability, $B_2$ and $B_4$. Thus $\#(\hat{\mathcal{A}}_\mu) = 2^{10} = 1024$.

(d) Illustrate Proposition 1.2.1 in the context of this exercise.

*Proof.* Consider $\mu$ as given in part (b), and let $B = \{1\}$. Consider extending $\mu$ to $\sigma[\hat{\mathcal{A}}_\mu \cup \{B\}]$ by defining $\mu(B) = a$ where $0 < a < 1/4$.

This is valid extension of $\mu$ for each $a \in (0, 1/4)$, but it is not unique since there are (uncountably) many choices for $a$.

5. PfS, Exercise 9.1.5, page 196.

   **Solution:** Easy differentiation:

$$
\begin{aligned}
-f_{W_n}(t) &= \frac{d}{dt} P(W_n > t) \\
&= \sum_{k=0}^{m-1} \left\{ k(\nu t)^{k-1} \frac{\nu e^{-\nu t}}{k!} - \frac{(\nu t)^k}{k!} \nu e^{-\nu t} \right\} \\
&= \nu e^{-\nu t} \left\{ \sum_{k=1}^{m-1} \left[ \frac{(\nu t)^{k-1}}{(k-1)!} - \frac{(\nu t)^k}{k!} \right] - 1 \right\} \\
&= \nu e^{-\nu t} \left\{ \left(1 - \frac{\nu t}{1}\right) + \left(\frac{\nu t}{1} - \frac{(\nu t)^2}{2!}\right) + \cdots + \left(\frac{(\nu t)^{m-2}}{(m-2)!} - \frac{(\nu t)^{m-1}}{(m-1)!}\right) - 1 \right\} \\
&= -\nu e^{-\nu t} \frac{(\nu t)^{m-1}}{(m-1)!}.
\end{aligned}
$$

   Thus (19) holds.

6. **Optional bonus problem:** Let $\Omega = \mathbb{Z}$ =the integers, and let $\mathcal{A}$ be the collection of subsets $A$ of $\mathbb{Z}$ so that $A$ or $A^c$ is finite. Let $\mu(A) = 0$ in the first case, and let $\mu(A) = 1$ in the second. Show that $\mu$ has no extension to $\sigma(\mathcal{A})$.

   **Solution:** Since $\mathbb{Z} = \cup_{n=1}^{\infty} \{-n, n\}$ with $\mu(\{-n, n\}) = 0$, countable additivity of $\mu$ on $\mathcal{A}$ would imply that $\mu(\mathbb{Z}) = \sum_{n=1}^{\infty} 0 = 0$. On the other hand, since $\mathbb{Z}^c = \emptyset$ is finite, $\mu(\mathbb{Z}) = 1$. Hence we conclude that $\mu$ is not countably additive on $\mathcal{A}$, and the Caratheodory extension theorem does not apply. Since $\mu$ is not countably additive on $\mathcal{A}$, there is certainly no extension which is countably additive on $\sigma(\mathcal{A})$.