Statistics 521, Midterm Exam Solutions

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1. (24 points). **Define** *three* of the following five terms:
(a) Convergence in probability of a sequence of random variables $\{X_n\}$.
(b) Convergence in distribution of a sequence of random variables $\{X_n\}$.
(c) $\liminf A_n$ for a sequence of events $\{A_n\}$.
(d) A uniformly integrable sequence of random variables $\{X_n\}$
(e) A *simple function* defined on a measurable space $(\Omega, \mathcal{A})$.

   Solution: See PfS, chapters 1-3.

2. (20 points). Give careful **statements** of *two* of the following four theorems or results:
(a) The monotone convergence theorem.
(b) The Helly-Bray theorem.
(c) Liapunov's inequality.
(d) The Caratheodory extension theorem.

   Solution: See PfS, chapters 1-3.

3. (30 points).
(a) Suppose that X is a non-negative measurable function on a measurable space $(\Omega, \mathcal{A})$. Give an explicit sequence of simple functions X_n satisfying $X_n \nearrow X$.
(b) Now suppose that $(\Omega, \mathcal{A}) = ((0,1), \mathcal{B}_{(0,1)})$, and that we give this measurable space the Lebesgue measure λ, which we call P since it is a probability measure on this $(\Omega, \mathcal{A})$. Suppose that $X(\omega) = \omega^{-1/2}$ for $\omega \in (0,1)$.
(b-1) For the simple functions X_n as given in (a), evaluate

$$\lim_{n\to\infty} \int X_n dP = \lim_{n\to\infty} E(X_n).$$

(b-2) Find the (induced) distribution function $F = F_X$ of X on $\mathbb{R}$.

   Solution: (a) From section 2.2 (and (2.2.10) in particular), for $X \geq 0$ we can take

$$X_n \equiv \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1\{\frac{k-1}{2^n} \leq X < \frac{k}{2^n}\} + n1_{[X\geq n]}.$$

Then $X_n(\omega) \nearrow X(\omega)$ for every ω.
(b1) By the monotone convergence theorem,

$$E(X_n) \to E(X) = \int_0^1 \omega^{-1/2} d\omega = \frac{\omega^{1/2}}{1/2}\Big|_0^1 = 2.$$

(b2) By definition

$$\begin{aligned} F(x) &= P(X \le x) = P(\{\omega : \omega^{-1/2} \le x\}) = P(\{\omega : \omega \ge x^{-2}\}) \\ &= \begin{cases} 1 - x^{-2}, & \text{for } x \ge 1, \\ 0, & \text{for } x \le 1. \end{cases} \end{aligned}$$

Also note that

$$E(X) = \int_0^\infty (1 - F(x))dx = \int_0^1 1dx + \int_1^\infty x^{-2}dx = 1 + 1 = 2.$$

4. (24 points). Let $X, Y \ge 0$ a.s. with $P(XY \ge 1) = 1$ and $P(\Omega) = 1$. Let $\mu_X = E(X)$, $\mu_Y = E(Y)$.
(a) Show that $\mu_X \cdot \mu_Y \ge 1$.
(b) Show that

$$(1 + \mu_X^2)^{1/2} \le E\{(1 + X^2)^{1/2}\} \le 1 + \mu_X.$$

Solution: (a) Since $XY \ge 1$ a.s. we have $1 = \sqrt{1} \le \sqrt{XY}$ a.s. and hence, by the Cauchy-Schwarz inequality,

$$1 \le E(\sqrt{XY}) \le \sqrt{E(X)E(Y)} = \sqrt{\mu_X \mu_Y},$$

and this gives the conclusion by squaring.
(b) Let $g(x) = (1 + x^2)^{1/2}$. Then we compute

$$\begin{aligned} g'(x) &= \frac{x}{\sqrt{1 + x^2}}, \\ g''(x) &= \frac{1}{\sqrt{1 + x^2}} + x(-1/2)(1 + x^2)^{-3/2}(2x) \\ &= \frac{1}{(1 + x^2)^{3/2}} > 0 \qquad \text{for } x \ge 0. \end{aligned}$$

Thus g is convex, and by Jensen's inequality

$$g(\mu_X) = g(E(X)) \le Eg(X),$$

which is the inequality on the left side in (b). On the other hand, for $x \ge 0$, $g(x) \le 1 + x$ for $x \ge 0$ (since $g(x)^2 = 1 + x^2 \le 1 + 2x + x^2 = (1 + x)^2$), and hence $Eg(X) \le E(1 + X) = 1 + E(X) = 1 + \mu_X$.

5. (30 points). Consider the sequence of random variables $\{X_n\}_{n \ge 1}$ with df's

$$F_n(x) = P(X_n \le x) = \begin{cases} (1 - n^{-2})x, & 0 \le x \le 1, \\ (1 - n^{-2}), & 1 \le x < n, \\ 1, & n \le x < \infty. \end{cases}$$

(a) Does $X_n \to_d$ "some" X? If so, what is the distribution function F of X?
(b) Does $\exp(\sin(\pi X_n/2)) \to_d$ something? If so, what is something?

(c) Compute EX_n^r for $r > 0$ and $n = 1, 2, \ldots$.
(d) For what values of $r > 0$ does $EX_n^r \to$ something finite?
(e) For what values of $r > 0$ is $\{X_n^r\}$ uniformly integrable?

Solution: (a) Note that

$$F_n(x) \to \left\{ \begin{array}{ll} x, & 0 \leq x \leq 1, \\ 1, & 1 \leq x < \infty \end{array} \right\} \equiv F(x),$$

the Uniform distribution on $[0, 1]$. So yes, $X_n \to_d X$ where $X \sim$ Uniform$[0, 1]$.
(b) The function $g(x) = \exp(\sin(\pi x/2))$ is continuous. Thus by the Mann-Wald or continuous mapping theorem $g(X_n) \to_d g(X)$ where $X \sim$ Uniform$[0, 1]$ as in (a).
(c) For any $r > 0$ and $n \geq 1$,

$$\begin{aligned} EX_n^r &= \int_0^\infty x^r dF_n(x) = (1 - n^{-2}) \int_0^1 x^r dx + n^{-2} n^r \\ &= (1 - n^{-2}) \frac{1}{r+1} + n^{-(2-r)}. \end{aligned}$$

(d) If $r < 2$, then $EX_n^r \to 1/(r + 1) < \infty$.
If $r = 2$, then $EX_n^r \to 1/(r + 1) + 1 = (r + 2)/(r + 1) < \infty$.
If $r > 2$, then $EX_n^r \to \infty$.
(e) Since $EX^r = \int_0^1 x^r dx = 1/(r+1)$ for all $r > 0$, we have $EX_n^r \to EX^r$ for $r < 2$. Thus by Vitali's theorem, $\{X_n^r\}_{n \geq 1}$ is uniformly integrable for $r < 2$.