Statistics 521, Problem Set 1, Solutions

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1. (a) Suppose that $\{\mathcal{A}_n\}$ is an increasing sequence of algebras, i.e. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ for all $n \ge 1$. Show that $\cup_{n=1}^{\infty} \mathcal{A}_n$ is an algebra.
   (b) Suppose that the $\mathcal{A}_n$ of (a) are $\sigma-$algebras. Show by constructing a counter-example that $\cup_{n=1}^{\infty} \mathcal{A}_n$ need not be a $\sigma-$algebra.

   Solution: (a) If $A \in \cup_{n=1}^{\infty} \mathcal{A}_n$, then $A \in \mathcal{A}_m$ for some m, and since $\mathcal{A}_m$ is an algebra, $A^c \in \mathcal{A}_m$. Hence $A^c \in \cup_{n=1}^{\infty} \mathcal{A}_n$. If $A, B \in \cup_{n=1}^{\infty} \mathcal{A}_n$, then $A \in \mathcal{A}_m$ for some m and $B \in \mathcal{A}_n$ for some n. Without loss we can assume that $m \le n$, and since $\mathcal{A}_m \subset \mathcal{A}_n$ it follows that $A, B \in \mathcal{A}_n$. Since $\mathcal{A}_n$ is an algebra, it follows that $A \cup B \in \mathcal{A}_n$, and hence that $A \cup B \in \cup_{n=1}^{\infty} \mathcal{A}_n$.
   (b) Take $\Omega = [0, 1]$. Let $\mathcal{A}_1 = \{\emptyset, \Omega\}$, $\mathcal{A}_2 = \sigma[\mathcal{A}_0, [0, 1/2)]$, ..., $\mathcal{A}_n = \sigma[\mathcal{A}_{n-1}, [0, 1 - 1/n)]$, ... Then $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ by construction, but $\cup_{n=1}^{\infty} \mathcal{A}_n$ is not a sigma field: if we let $A_k = [0, 1 - 1/k)$ for each $k = 1, 2, \ldots$, then $A_k \in \cup_{n=1}^{\infty} \mathcal{A}_n$ since $A_k \in \mathcal{A}_k$ by construction, but $[0, 1) = \cup_{k=1}^{\infty} A_k \notin \cup_{n=1}^{\infty} \mathcal{A}_n$.

2. Write out a proof of Proposition 1.1(b), PfS, page 3: There exists a minimal field, $\sigma-$field, or monotone class generated by (or containing) any specified class $\mathcal{C}$ of subsets of Ω.

   Solution: By proposition 1.1.1(a), arbitrary intersections of fields, $\sigma-$fields, or monotone classes are again fields, $\sigma-$fields, or monotone classes. Hence

$$\phi[\mathcal{C}] \equiv \cap\{\mathcal{A}_\alpha : \mathcal{A}_\alpha \ \text{ is a } \ \text{-field of subsets of } \ \Omega \ \text{ for which } \ \mathcal{C} \subset \mathcal{A}_\alpha\}$$

   is again a field, and it is the smallest such field: if $\mathcal{D}$ is the minimal field containing $\mathcal{C}$ so that $\mathcal{D} \subset \phi[\mathcal{C}]$, then we also have $\phi[\mathcal{C}] \subset \mathcal{D}$ by construction of $\phi[\mathcal{C}]$, and hence $\phi[\mathcal{C}] = \mathcal{D}$. The argument is the same for $\sigma-$fields and monotone classes with $\phi[\mathcal{C}]$ replaced by $\sigma[\mathcal{C}]$ and $\text{mon}[\mathcal{C}]$ respectively.

3. PfS, Exercise 1.1.1, page 4. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote two collections of subsets of the set Ω. If $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$ and $\mathcal{C}_2 \subset \sigma[\mathcal{C}_1]$, then $\sigma[\mathcal{C}_1] = \sigma[\mathcal{C}_2]$.

   Solution: Since $\mathcal{C}_1 \subset \sigma[\mathcal{C}_2]$, it follows immediately that $\sigma[\mathcal{C}_1] \subset \sigma[\sigma[\mathcal{C}_2]] = \sigma[\mathcal{C}_2]$. By a symmetric argument $\sigma[\mathcal{C}_2] \subset \sigma[\mathcal{C}_1]$. Hence $\sigma[\mathcal{C}_2] = \sigma[\mathcal{C}_1]$.

4. PfS, Exercise 1.1.2, page 8: We always have $\mu(\liminf A_n) \le \liminf \mu(A_n)$, while $\limsup \mu(A_n) \le \mu(\limsup A_n)$ holds if $\mu(\Omega) < \infty$.

   Solution: First note that $\liminf A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k = \cup_{n=1}^{\infty} B_n$ where $B_n = \cap_{k=n}^{\infty} A_k$ is $\uparrow$ since $B_n = \cap_{k=n}^{\infty} A_k \subset \cap_{k=n+1}^{\infty} A_k = B_{n+1}$ for all n. Hence by Proposition 1.2(a),

$$\begin{aligned}
\mu(\liminf A_n) &= \mu(\cup_{n=1}^{\infty} B_n) \\
&= \lim_{n\to\infty} \mu(B_n) \\
&= \lim_{n\to\infty} \mu(\cap_{k=n}^{\infty} A_k) \\
&\le \lim_{n\to\infty} \inf_{m\ge n} \mu(A_m) \\
&= \liminf \mu(A_n)
\end{aligned}$$

   since $\cap_{k=n}^{\infty} A_k \subset A_m$ for each $m \ge n$ so that

$$\mu(\cap_{k=n}^{\infty} A_k) \le \mu(A_m)$$

   for each $m \ge n$ and also $\mu(\cap_{k=n}^{\infty} A_k) \le \inf_{m\ge n} \mu(A_m)$.
   Similarly, $\limsup A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = \cap_{n=1}^{\infty} B_n$ where $B_n = \cup_{k=n}^{\infty} A_k$ is $\downarrow$ since $B_n = \cup_{k=n}^{\infty} A_k \supset \cup_{k=n+1}^{\infty} A_k = B_{n+1}$. Thus by Proposition 1.1.2(b), if $\mu(\Omega) < \infty$,

$$\begin{aligned}
\mu(\limsup A_n) &= \mu(\cap_{n=1}^{\infty} B_n) \\
&= \lim_{n\to\infty} \mu(B_n) \\
&= \lim_{n\to\infty} \mu(\cup_{k=n}^{\infty} A_k) \\
&\ge \lim_{n\to\infty} \sup_{m\ge n} \mu(A_m) \\
&= \limsup \mu(A_n)
\end{aligned}$$

   since $\cup_{k=n}^{\infty} A_k \supset A_m$ for each $m \ge n$ so that

$$\mu(\cup_{k=n}^{\infty} A_k) \ge \mu(A_m)$$

   for each $m \ge n$ and also $\mu(\cup_{k=n}^{\infty} A_k) \ge \sup_{m\ge n} \mu(A_m)$.

5. PfS, Exercise 9.1.4, page 182: if $np_n \to \lambda > 0$, then

$$P(T_n = k) = \binom{n}{k} p_n^k (1 - p_n)^{n-k} \to \frac{\lambda^k}{k!} \exp(-\lambda) = P(Y = k)$$

where $Y \sim Poisson(\lambda)$.

Solution:

$$\begin{aligned} P(T_n = k) &= \binom{n}{k} p_n^k (1 - p_n)^{n-k} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k! n^k} (np_n)^k (1 - \frac{np_n}{n})^{n-k} \\ &\to \frac{1}{k!} \lambda^k e^{-\lambda} \end{aligned}$$

since $(1 + x_n/n)^n \to e^x$ if $x_n \to x$.