

Statistics 521, Problem Set 8

Wellner; 11/14/2007

Reading:

- Shorack, PfS, Chapter 7, pages 121 - 136; Chapter 8, pages 165 - 171.

Due: Wednesday, November 21, 2007.

1. PfS Exercise 7.4.3, page 131. Prove just the parts of these formulas involving F , not the parts involving F^{-1} . You may also use Fubini's theorem directly.
2. Prove the two formulas in (17), PfS page 131.
3. PfS, Exercise 5.1.4, page 91.
Let $\mathcal{X} = [0, 1]$, $\mathcal{Y} = (1, \infty)$ both equipped with the Borel sets and Lebesgue measure. Let $f(x, y) = e^{-xy} - 2e^{-2xy}$. Show that
 - (i) $\int_0^1 (\int_1^\infty f(x, y) dy) dx = \int_0^1 x^{-1} (e^{-x} - e^{-2x}) dx$ exists and is > 0 .
 - (ii) $\int_1^\infty (\int_0^1 f(x, y) dx) dy = \int_1^\infty y^{-1} (e^{-2y} - e^{-y}) dy$ exists and is < 0 .
4. Show that $f(x, y) = e^{-xy} \sin(x)$ is integrable with respect on Lebesgue measure on R^2 in the strip $0 < x < a$, $0 < y$. Perform the double integral in the two different orders to find that

$$\int_0^a \frac{\sin(x)}{x} dx = \frac{\pi}{2} - \cos(a) \int_0^\infty \frac{e^{-ay}}{1+y^2} dy - \sin(a) \int_0^\infty \frac{ye^{-ay}}{1+y^2} dy.$$

Use the inequality $1 + y^2 \geq 1$ to obtain the bound

$$\left| \int_0^a \frac{\sin(x)}{x} dx - \frac{\pi}{2} \right| \leq \frac{2}{a}.$$

Letting $a \rightarrow \infty$ yields $\int_0^\infty x^{-1} \sin(x) dx = \pi/2$.

5. **Optional bonus problem:** Suppose that $\phi : (0, 1) \rightarrow R^+$ satisfies:

- (i) $\phi(1) = 0$,
- (ii) $\phi' < 0$,
- (iii) $\phi'' > 0$.
- (iv) $\phi(u) \rightarrow \infty$ as $u \downarrow 0$.

For example: (a) $\phi(u) = (u^{-\theta} - 1)/\theta$, $0 < \theta < \infty$; (b) $\phi(u) = (-\log u)^{1/\theta}$, $0 < \theta < 1$; (c) $\phi(u) = \log(\frac{1-\theta}{1-\theta^u})$, $0 < \theta < \infty$, $\theta \neq 1$.

Define $C : (0, 1] \times (0, 1] \rightarrow [0, 1]$ by $C(u, v) = \phi^{-1}(\phi(u) + \phi(v))$.

A. Show that C is the joint distribution function of a pair of random variables (U, V) which have the property that the marginal distributions of U and V are Uniform(0, 1).

B. Show that under (i) - (iv) (U, V) have a joint density function $c(u, v)$ and calculate it; i.e. show that $C(u, v) \ll \text{Lebesgue measure on } [0, 1]^2$.

C. What if $\lim_{u \rightarrow 0} \phi(u) = \phi(0) < \infty$? Does (U, V) have a joint density function? Show that if $\lim_{u \rightarrow 0} \phi(u) = \phi(0) < \infty$, then C is still a distribution function, but with one component C_{ac} with a density function c_{ac} ("ac" for "absolutely continuous with respect to Lebesgue measure λ_2 on R^2), and a component which is singular with respect to λ_2 . Show that the singular component has probability $p = -\frac{\phi(0)}{\phi'(0)}$ on the curve given by: $\phi(u) + \phi(v) = \phi(0)$.