Statistics 521, Problem Set 7
Wellner; 11/07/2007

Reading: Shorack, PfS, Chapter 5, pages 87 - 102;
Chapter 7, pages 121 - 132.

Reminder: Midterm Exam, Friday, November 9.
Due: Wednesday, November 14, 2007.

1. Let μ be a sigma-finite measure and let ν be a finite measure on $(\Omega, \mathcal{A})$. Set $\phi = \mu - \nu$; i.e. define $\phi : \mathcal{A} \to (-\infty, \infty]$ by $\phi(A) = \mu(A) - \nu(A)$.
(a) Show that ϕ is a signed measure.
(b) Show that
$$\phi(A) = \int_A (f - g) d(\mu + \nu)$$
for some measurable functions f and g, $g \in \mathcal{L}_1(\mu + \nu)$. Thus ϕ can be written in the canonical form of the signed measure discussed in example 4.1.1 and Problem Set 6, problem 5.
(c) Apply the results of Problem Set 6, problem 5 (i.e. PfS Exercise 4.1.2, page 67) to ϕ: compute ϕ^+, ϕ^-, $|\phi|$, and $|\phi|(\Omega)$, assuming for the latter that μ is also a finite measure.

2. For probability measures P and Q on $(\Omega, \mathcal{A})$, define
$$d_{TV}(P, Q) = \sup_{A \in \mathcal{A}} |P(A) - Q(A)| .$$
You showed in problem 2, problem set #6 that $d_{TV}(P, Q) = (1/2) \int |p - q| d\mu$ for any measure μ dominating both P and Q; i.e. $P << \mu$, $Q << \mu$.
(a) Show that $d_{TV}(P, Q)$ does not depend on the choice of μ.
(b) Use the results of problem 1, part (c) to show that $d_{TV}(P, Q) = (1/2)|P - Q|(\Omega)$.

3. Suppose that ϕ is a sigma-finite signed measure and $X \in L_1(|\phi|)$. The integral $\int X d\phi$ is defined by
$$\int X d\phi = \int X d\phi^+ - \int X d\phi^- .$$
Show that $|\int X d\phi| \le \int |X| d|\phi|$.

4. PfS, Exercise 4.2.3, page 73: Flip a coin. If heads results, let X be a Uniform$(0, 1)$ random variable; if tails results, let X be a Poisson(λ) random variable. The resulting distribution of X on $\mathbb{R}$ is labeled ϕ.
   (a) Let μ denote Lebesgue measure on $\mathbb{R}$. Find the Lebesgue decomposition of ϕ with respect to μ; that is, write $\phi = \phi_{ac} + \phi_s$.
   (b) Let μ be counting measure on $\{0, 1, 2, \ldots\}$. Find the Lebesgue decompositionn of ϕ with respect to μ.

5. PfS, Exercise 4.4.3, page 84: Let F be $\nearrow$, right-continuous and bounded on $\mathbb{R}$ with $F(-\infty) = 0$. Define μ_F via $\mu_F((a, b]) = F(b) - F(a)$ for all $a < b$. Show that $\mu_F << \lambda$ if and only if F is an absolutely continuous function on $\mathbb{R}$.