Statistics 521, Problem Set 5

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Reading: Shorack, PfS, Chapter 4, sections 4.1 - 4.4, pages 67 - 85.

Due: Wednesday, October 31, 2007.

1. PfS, Exercise 3.4.2, page 48: Show that $\rho = 1$ if and only if $X - \mu_X = a(Y - \mu_Y)$ for some $a > 0$; and $\rho = -1$ if an only if $X - \mu_X = a(Y - \mu_Y)$ for some $a < 0$. Thus ρ measure linear dependence, not dependence.

2. PfS, Exercise 3.4.3, page 48: (Littlewood's inequalities) Let $\mu_r \equiv E|X|^r$. For $r \ge s \ge t \ge 0$ we have $\mu_r^{s-t}\mu_t^{r-s} \ge \mu_s^{r-t}$.

3. Suppose that $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. random variables with $P(\epsilon_i = \pm 1) = 1/2$, and let $a_i \in R$, $i = 1, \ldots, n$. Prove the Khintchine inequalities for the case $p = 1$: for some constants A_p and B_p

$$A_p(\sum_{i=1}^n a_i^2)^{1/2} \le (E|\sum_{i=1}^n a_i\epsilon_i|^p)^{1/p} \le B_p(\sum_{i=1}^n a_i^2)^{1/2} .$$

   Hint: The inequality on the right side is easy. Use the previous exercise to prove the inequality on the left side.

4. PfS, Exercise 3.5.3, page 55: Consider a probability measure P. (a) Let $Y \ge 0$ have df F. Then $EY = \int_0^\infty P(Y \ge y)dy = \int_0^\infty [1 - F(y)]dy$ will be shown to hold in (7.4.11) below. For now, we shall just use this fact. (The claimed formula can also be established for simple functions by summing by parts; and then the full claim follows from the MCT. This cumbersome proof is possible now. It constitutes this exercise.) (b) use the result of 9a) to show that for $Y \ge 0$ and $\lambda \ge 0$ we have

$$\int_{[Y \ge \lambda]} Y dP = \lambda P(Y \ge \lambda) + \int_\lambda^\infty P(Y \ge y)dy.$$

   (c) Suppose there is a $Y \in \mathcal{L}_1$ such that $P(|X_n| \ge y) \le P(Y \ge y)$ for all $y > 0$ and all $n \ge 1$. Then use (b) to show that $\{X_n : \ n \ge 1\}$ is uniformly integrable.

5. (a) Show that if $|X_n| \le Y$ and Y is integrable, then $\{X_n\}$ is uniformly integrable.
(b) Let $U \sim$Uniform$(0,1)$, and let $X_n \equiv (n/\log n)1_{[0,1/n]}(U)$ for $n \ge 3$. Show that $\{X_n\}$ is uniformly integrable and $\int X_n dP \to 0$ even though they are not dominated by any integrable rv Y.
(c) Let $Z_n = n1_{[0,1/n]}(U) - n1_{[1/n,2/n]}(U)$. Show that $\{Z_n\}$ is not uniformly integrable, but that $\int Z_n dP \to 0$.

6. **Optional bonus problem:** PfS, Exercise 3.4.6, page 50 (qualified by "for all $\epsilon \ge 1$"): Let $T \sim$ Binomial(n,p), so $P(T=k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $0 \le k \le n$. The measure associated with T has mean np and variance $np(1-p)$. Then use inequality 4.6 with $g(x) = \exp(rx)$ and $r > 0$ to show that

$$P(T/n \ge p\epsilon) \le \exp(-nph(\epsilon)), \qquad \text{where } h(\epsilon) \equiv \epsilon(\log(\epsilon) - 1) + 1$$

for each $\epsilon > 1$. [Hint: It helps to use $T \stackrel{d}{=} \sum_1^n X_i$ where $X_i \sim$ Bernoulli(p) are independent, and then apply Theorem 8.1.1.]