

## Statistics 521, Problem Set 4

Wellner; 10/17/2007

### Reading:

- Shorack, PfS, Chapter 3, sections 3.4 - 3.5, pages 46 - 63.

**Reminder:** Make-up lectures on Wednesday 10/17 and Wednesday 10/27, 12:30 - 1:20 in Loew 222.

**Due:** Wednesday, October 24, 2007.

1. PfS, Exercise 2.3.3, page 32: Show that  $X_n \rightarrow_{\mu} X$  if and only if  $X_m - X_n \rightarrow_{\mu} 0$ . [Hint: adapt the proof (14).]
2. PfS, Exercise 2.3.4, page 32: (a) Suppose that  $\mu(\Omega) < \infty$  and  $g$  is continuous a.e.  $\mu_X$  (that is,  $g$  is continuous except perhaps on a set of  $\mu_X$  measure 0). Then  $X_n \rightarrow_{\mu} X$  implies that  $g(X_n) \rightarrow_{\mu} g(X)$ .  
(b) Let  $g$  be uniformly continuous on the real line. Then  $X_n \rightarrow_{\mu} X$  implies that  $g(X_n) \rightarrow_{\mu} g(X)$ . (Here  $\mu(\Omega) = \infty$  is allowed.)
3. PfS, Exercise 3.2.1, page 42: Show that  $X \geq 0$  and  $\int X d\mu = 0$  implies  $\mu(\{X > 0\}) = 0$ .
4. PfS, Exercise 3.2.2, page 42: Show that

$$\int_A X d\mu = \begin{cases} = 0, \\ \geq 0, \end{cases} \quad \text{for all } A \in \mathcal{A} \text{ implies } X = \begin{cases} = 0 \text{ a.e.}, \\ \geq 0 \text{ a.e.} \end{cases}$$

5. (See example 1.1, page 123.) The Cantor singular distribution function  $F$  is the function  $F : [0, 1] \rightarrow [0, 1]$  defined as follows:  $F(x) = 1/2$  for  $x \in (1/3, 2/3)$ ;  $F(x) = 1/4$  for  $x \in (1/9, 2/9)$  and  $F(x) = 3/4$  for  $x \in (7/9, 8/9)$ ;  $\dots$ ;  $F(x) = 1/2^n, 3/2^n, 5/2^n, \dots$  on the successive intervals removed from  $C_{n-1}$  in the construction of  $C_n$ . Thus  $F$  is defined on the open set  $[0, 1] \setminus C$ , is nondecreasing, and has values in  $[0, 1]$ . Extend it to all of  $[0, 1]$  by letting  $F(0) = 0$ , and setting

$$F(x) \equiv \sup\{F(y) : t \in [0, 1] \setminus C \text{ and } y < x\}$$

for  $x \in C$  and  $x \neq 0$ . Show that  $F$  is non-decreasing and continuous with  $F(0) = 0$  and  $F(1) = 1$ . Because  $F$  is continuous, its range is all of  $[0, 1]$ . Now the inverse (or quantile) function  $F^{-1}$  of  $F$  defined by

$$F^{-1}(y) \equiv \inf\{x \in [0, 1] : F(x) \geq y\}$$

is one-to-one (injective) and  $F^{-1}([0, 1]) \subset C$ . Show that  $F^{-1}$  is Borel-measurable.