

Statistics 522, Problem Set 2

Wellner; 1/23/2008

Reading:

Shorack, PfS, Chapter 11, Sections 7-8, pages 312 - 318;

Shorack, PfS, Chapter 13, Sections 1-7, pages 367 - 390.

Due: Wednesday, January 30, 2008.

1. (Symmetry and conditional expectation). Let X_1, X_2, \dots be i.i.d. random variables with the same distribution as X where $E|X| < \infty$. Let $S_n \equiv X_1 + \dots + X_n$, and define

$$\mathcal{G}_n \equiv \sigma [S_n, S_{n+1}, \dots] = \sigma [S_n, X_{n+1}, X_{n+2}, \dots].$$

Show that $E(X_1|\mathcal{G}_n) = E(X_1|S_n) = n^{-1}S_n$ almost surely. [Hint: Note that $\sigma [X_{n+1}, X_{n+2}, \dots]$ is independent of $\sigma [X_1, S_n]$, and use symmetry to show that $E(1_{[S_n \in B]}X_1) = E(1_{[S_n \in B]}X_2) = \dots = E(1_{[S_n \in B]}X_n)$.]

2. (An application of the WLLN: Inversion of Laplace transforms) Let P be a probability measure on the Borel subsets of $[0, \infty)$, and define its Laplace transform by $\varphi(t) = \int_0^\infty e^{-tx} dP(x)$ for $t \in [0, \infty)$. Widder's inversion formula for P from φ is:

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\lfloor nz \rfloor} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = P([0, z]) \quad (1)$$

Show that (1) holds via the following steps:

- (a) Differentiation of the integral k times shows that

$$\varphi^{(k)}(t) = \int_0^\infty (-x)^k e^{-tx} dP(x).$$

- (b) Setting $t = n$, letting $z > 0$, multiplying across by $(-1)^k n^k / k!$, and summing on k yields

$$\sum_{k=0}^{\lfloor nz \rfloor} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = \int_0^\infty \sum_{k=0}^{\lfloor nz \rfloor} e^{-nx} \frac{(nx)^k}{k!} dP(x) \quad (2)$$

where $e^{-nx}(nx)^k/k! = P(S_n = k)$ and $S_n = Y_1 + \dots + Y_n$ where Y_1, Y_2, \dots are i.i.d. $\text{Poisson}(x)$.

(c) Use the weak law of large numbers and (2) to show that (1) holds.

3. PfS, Exercise 8.6.1, page 188.
4. PfS, Exercise 8.6.2, page 189.
5. Suppose that $X, Y \in L_1(\Omega, \mathcal{F}, P)$ and that $E(Y|X) = X$ a.s. and $E(X|Y) = Y$ a.s. Prove that $P(X = Y) = 1$. (See e.g. Exercise 9.2, Williams, *Probability with Martingales*, page 231.)