Statistics 521, Practice Final Exam

Wellner

1. (10 points). **Define** the following terms:
 (a) Independent $\sigma-$fields; independent random variables.
 (b) The tail $\sigma-$field of a sequence $X_1, X_2, \ldots$.

2. State the following results:
 (a) The Radon - Nikodym theorem
 (b) A product measure $\mu \times \nu$.
 (c) Kolmogorov's 0-1 law.
 (d) The first Borel-Cantelli lemma.
 (e) Any decomposition theorem for measures.

3. (20 points) Let $X_1, \ldots, X_n, \ldots$ be independent, and let $S_n = X_1 + \cdots + X_n$, $n \geq 1$.
 (a) (5 points) What does Chebychev's inequality say about the S_n's?
 (b) (5 points) What does Kolmogorov's inequality say about the S_n's? Draw a picture to illustrate the difference or key distinction.
 (c) (10 points) Prove Kolmogorov's inequality.

4. Suppose that $\{A_n\}$ and $\{B_n\}$ are sequences of events with $P(A_n \text{ i.o.}) = 1$ and $P(B_n^c \text{ i.o.}) = 0$. Then is

$$P(A_n \cap B_n \text{ i.o.}) = \begin{cases} 0 \\ 1 \\ \text{undetermined?} \end{cases}$$

5. (20 points) Suppose that $X_1, X_2, \ldots$ are independent with $P(X_n = 1/2^n) = 1/2 = P(X_n = 0)$. Prove that $\sum_{k=1}^n X_k \to_{a.s.}$ some rv X. What are the mean and variance of X; at least indicate your intuition carefully.

6. (30 points) Suppose that X_n, $n = 1, 2, \ldots$ are independent rv's with df's F_n given by

$$1 - F_n(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ (1+x)^{-a_n}, & \text{if } x \geq 0 \end{cases}$$

where a_n is a sequence of positive numbers.
(a) Find the distribution of $M_n \equiv \min_{1 \le k \le n} X_k$.
(b) Find a condition on the a_n's which will guarantee $M_n \rightarrow_d$ some M_0 which is not degenerate at 0, and give the limiting distribution df. Give an example of such a sequence a_n.
(c) find a condition on the a_n's which will guarantee $nM_n \rightarrow_d$ some M_0, and give the limiting distribution. Give an example of such a sequence a_n.
(d) For sequences a_n such that $M_n \rightarrow_d M_0$ as in (b), construct a sequence M_n^* such that M_n^* has the same distribution as M_n for each $n \ge 0$, but $M_n^* \rightarrow_{a.s.} M_0^*$.
(e) For sequences a_n such that $M_n \rightarrow_d M_0$ as in (b), does $\log M_n \rightarrow_d$? Why? (Name a result that does this if there is one.)
(f) Suppose that $a_n = a$ (constant) for all n, so the X_n are i.i.d. (as X say). For what values of a is $EX^r < \infty$? (You may use a formula which was derived in recent HW.)

7. (20 points) Suppose P_1 is the measure with density $p_1(x) = (1/2)1_{[0,2]}(x)$ with respect to Lebesgue measure, and P_2 is the measure with density $p_2(x) = (1/2)1_{[1,3]}(x)$; i.e. P_1 is Uniform$(0,2)$ and P_2 is Uniform$(1,3)$.
(a) Is $P_1 \ll P_2$? Why or why not?
(b) Give the Lebesgue decomposition of P_1 with respect to P_2.
(c) Is $P_1 \ll P_1 + P_2$? Why or why not?
(d) Give the Lebesgue decomposition of P_1 with respect to $P_1 + P_2$.