Statistics 521, Midterm Exam

Wellner; 11/9/98

1. (18 points). **Define** *three* of the following five terms:
   (a) Convergence a.e. of a sequence of measurable functions $\{X_n\}$.
   (b) Convergence in rth mean of a sequence of random variables $\{X_n\}$.
   (c) $\limsup A_n$ for a sequence of events $\{A_n\}$.
   (d) A uniformly integrable sequence of random variables $\{X_n\}$
   (e) A $\pi-$system and a $\lambda-$system.

2. (24 points). Give careful **statements** of *three* of the following four theorems or results:
   (a) Fatou's lemma.
   (b) The Caratheodory extension theorem.
   (c) Vitali's theorem.
   (d) The $\pi - \lambda$ theorem.

3. (25 points). Suppose that $U \sim \text{Uniform}(0,1)$, and let $X_n \equiv n a_n 1_{[0,1/n]}(U)$ for some sequence of real numbers a_n.
   (a) Compute $E(X_n)$.
   (b) Show that $X_n \to_p 0$ for any sequence $\{a_n\}$.
   (c) Give conditions on a_n which make $\{X_n\}$ uniformly integrable.
   (d) Give an example of a sequence a_n for which X_n is uniformly integrable, but there is no integrable dominating function Y for the family $\{X_n\}$.
   (e) Give an example of a sequence a_n for which $E(X_n) \to \infty$ as $n \to \infty$, and hence $\{X_n\}$ is *not* uniformly integrable.

4. (24 points). Let $(\Omega, \mathcal{A}) = ([0,1], \mathcal{B}_{[0,1]})$, $\mathcal{A}_0 = \{\emptyset, [0,1]\}$, and $\mathcal{A}_1 = \sigma[\{\emptyset, [0,1/3], (1/3,2/3], (2/3,1]\}]$. Consider $X_i(\omega)$ $i = 1,2$ defined by $X_1(\omega) = 2\omega$, $X_2(\omega) = (\pi/2)1_{[0,1/5]}(\omega) + 3 \cdot 1_{[3/4,1]}$.
   (a) Is X_1 $\mathcal{B} - \mathcal{A}_0$ measurable?
   (b) Find $X_2^{-1}(\mathcal{B})$. Is it a sigma - field?
   (c) Describe the collection of random variables that are $\mathcal{A}_1$ measurable.

5. (22 points). Let $\Omega = \{a, b, c, d, e\}$, and consider the sigma-field

$$\mathcal{A} = \{\emptyset, \{c\}, \{a, e\}, \{b, d\}, \{a, c, e\}, \{b, c, d\}, \{a, b, d, e\}, \{a, b, c, d, e\}\}\,.$$

Define P on $(\Omega, \mathcal{A})$ by $P(\{a, e\}) = 0$, $P(\{c\}) = .3$, $P(\{b, d\}) = .7$.
(a) Let $(\Omega, \widehat{\mathcal{A}}, \widehat{P})$ denote the completion of $(\Omega, \mathcal{A}, P)$. How many sets are in the sigma-field $\widehat{\mathcal{A}}$?
(b) How many sets are in the sigma-field $\mathcal{A}_1$ of *all* subsets of Ω?