The $\pi - \lambda$ Theorem
and Uniqueness of Extensions

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Definition. (Kallenberg, p. 2; Williams, p. 193). A collection of subsets $\mathcal{D}$ of a set Ω is called a $\lambda-$system (or sometimes a $d-$system) if it contains Ω and is closed under monotone increasing limits and proper differences: $\Omega \in \mathcal{D}$, $A, B \in \mathcal{D}$ with $B \subset A$ implies $A \setminus B \in \mathcal{D}$, and $A_n \in \mathcal{D}$, $A_n \uparrow A$ implies $A \in \mathcal{D}$.

Remark: Williams allows $B \subseteq A$ in his definition of a $d-$system.

Proposition. A collection $\mathcal{A}$ of subsets of Ω is a $\sigma-$field if and only if $\mathcal{A}$ is both a $\pi-$system and a $\lambda-$system.

Proof. Homework.

Lemma 1. (Sierpinski, Dynkin). If $\mathcal{C}$ is a $\pi-$system contained in a $\lambda-$system $\mathcal{D}$, then $\sigma(\mathcal{C}) \subset \mathcal{D}$.

Proof. Let $\lambda(\mathcal{C})$ be the smallest $\lambda-$ system containing $\mathcal{C}$. It clearly suffices to show that $\sigma(\mathcal{C}) = \lambda(\mathcal{C})$. By the proposition it suffices to show that $\lambda(\mathcal{C})$ is a $\pi-$system, since then it is also a $\sigma-$field containing $\mathcal{C}$. The proof of this goes in two steps:

Step 1: Let $\mathcal{D}_1 \equiv \{B \in \lambda(\mathcal{C}) : \ B \cap C \in \lambda(\mathcal{C}) \ \text{ for all } \ C \in \mathcal{C}\}$. Because $\mathcal{C}$ is a $\pi-$system, $\mathcal{D}_1 \supset \mathcal{C}$. Furthermore, $\mathcal{D}_1$ inherits the $\lambda-$system structure from $\lambda(\mathcal{C})$. (Proof: HW!). Thus $\mathcal{D}_1$ is a $\lambda-$system which contains $\mathcal{C}$ and hence $\mathcal{D}_1 = \lambda(\mathcal{C})$.

Step 2: Let $\mathcal{D}_2 \equiv \{A \in \lambda(\mathcal{C}) : \ A \cap B \in \lambda(\mathcal{C}) \ \text{ for all } \ B \in \lambda(\mathcal{C})\}$. As in step 1, $\mathcal{C} \subset \mathcal{D}_2$. Moreover, just as in step 1, $\mathcal{D}_2$ inherits the $\lambda-$system structure from $\lambda(\mathcal{C})$, and hence $\mathcal{D}_2 = \lambda(\mathcal{C})$. But this just says that $\lambda(\mathcal{C})$ is a $\pi-$system. $\square$

Lemma 2. (Dynkin). Suppose that $\mathcal{C}$ is a $\pi-$system, and let $\sigma(\mathcal{C}) \equiv \mathcal{A}$. Suppose that μ_1, μ_2 are two measures on $(\Omega, \mathcal{A})$ with $\mu_1(\Omega) = \mu_2(\Omega) < \infty$ and $\mu_1 = \mu_2$ on $\mathcal{C}$ (i.e. $\mu_1 = \mu_2$ on the $\overline{\pi}-$system $\overline{\mathcal{C}} \equiv \mathcal{C} \cup \{\Omega\}$). Then $\mu_1 = \mu_2$ on $\mathcal{A} = \sigma(\mathcal{C})$.

Proof. Let $\mathcal{D} = \{A \in \mathcal{A} : \mu_1(A) = \mu_2(A)\}$. Then $\mathcal{D}$ is a $\lambda-$system on Ω: $\Omega \in \mathcal{D}$ is given. If $A, B \in \mathcal{D}$, then

$$\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \setminus A)\,,$$

so that $B \setminus A \in \mathcal{D}$. Moreover, if $A_n \in \mathcal{D}$ and $A_n \uparrow A$, then by Proposition 1.2 (i),

$$\mu_1(A) = \lim_n \mu_1(A_n) = \lim_n \mu_2(A_n) = \mu_2(A)\,,$$

so that $A \in \mathcal{D}$. Thus $\mathcal{D}$ is a $\lambda-$system.

Since $\mathcal{D}$ is a $\lambda-$system and $\mathcal{C} \subseteq \mathcal{D}$ by hypothesis, Lemma 1 shows that $\mathcal{A} = \sigma(\mathcal{C}) \subset \mathcal{D}$, and the result follows. $\square$