Two Inequalities for Mills' Ratio

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Suppose that $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ is the standard normal density, and let Φ denote the standard normal distribution function:

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt\,.$$

Inequality 1. For all $x > 0$ we have

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x) \le 1 - \Phi(x) \le \frac{1}{x}\phi(x)\,.$$

Proof. The proof of the inequality on the right side is easy:

$$1 - \Phi(x) = \int_x^\infty \phi(y)dy \le \int_x^\infty \frac{y}{x}\phi(y)dy = \frac{1}{x}\phi(x)$$

since $\phi'(y) = -y\phi(y)$. The inequality on the left takes a bit more work: first note that

$$(1+u^2)^{-1}\exp(-x^2(1+u^2)/2) = \int_{x^2/2}^\infty \exp(-v(1+u^2))\,dv.$$

Integrating this identity from $-\infty$ to ∞ and inverting the order of integration on the right side gives

$$\begin{aligned}
\int_{-\infty}^\infty (1+u^2)^{-1}\exp(-x^2(1+u^2)/2)\,du &= \int_{x^2/2}^\infty \int_{-\infty}^\infty \exp(-v(1+u^2))\,du\,dv \\
&= \sqrt{2\pi}\int_{x^2/2}^\infty (2v)^{-1/2}\exp(-v)\,dv \\
&= \sqrt{2\pi}\int_x^\infty \exp(-u^2/2)\,du \\
&= 2\pi(1-\Phi(x))
\end{aligned}$$

after changing variables. Hence we find that

$$\begin{aligned} 1-\Phi(x) &= (2\pi)^{-1}\int_{-\infty}^{\infty}(1+u^2)^{-1}\exp(-x^2(1+u^2)/2)\,du \\ &= \phi(x)(2\pi)^{-1/2}\int_{-\infty}^{\infty}(1+u^2)^{-1}\exp(-u^2x^2/2)\,du, \end{aligned}$$

and therefore that

$$R(x) \equiv (1-\Phi(x))/\phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}(1+u^2)^{-1}\exp(-u^2x^2/2)\,du.$$

If we change variables in the integral on the right side, letting $t = x^2u^2/2$, we find that

$$R(x) = x(2\sqrt{\pi})^{-1}\int_0^{\infty}\frac{t^{-1/2}\exp(-t)}{t+(1/2)x^2}\,dt\,.$$

Now we expand the denominator inside the integral:

$$\frac{1}{t+(1/2)x^2} = \frac{2}{x^2}\left(1-\frac{2t}{x^2}+\frac{4t^2}{x^4}-...\right)\;;$$

plugging this into the integral, integrating term by term, and expressing the resulting integrals in terms of gamma functions yields

$$R(x) = \frac{1}{x}-\frac{1}{x^3}+\frac{1\cdot 3}{x^5}-\cdots(-1)^j\frac{1\cdot 3\cdot 5\cdots(2j-1)}{x^{2j+1}}+R_j(x)$$

where

$$|R_j(x)| = \frac{2x^{-(2j+1)}}{2\sqrt{\pi}}\int_0^{\infty}\frac{t^{2j+2}\exp(-t)t^{-1/2}}{t+x^2}dt$$

has $|R_j(x)| < x^{-(2j+1)}1\cdot 3\cdot 5\cdots(2j-1)$, and hence is smaller than the last term taken into account . In particular this shows that

$$R(x) \geq \frac{1}{x}-\frac{1}{x^3}\,.$$

It also shows (again) that $R(x) \leq 1/x$. This proof is from Kendall and Stuart, *Advanced Theory of Statistics, Volume I*, pages 136 and 137.

Inequality 2. For all $x > 0$ we have

$$\frac{2}{x + \sqrt{x^2 + 4}}\phi(x) \leq 1 - \Phi(x) \leq \frac{2}{x + \sqrt{x^2 + 2}}\phi(x) \,.$$

Proof. (Sketch). Set

$$f(x) \equiv \frac{2}{x + \sqrt{x^2 + 4}}, \qquad\qquad g(x) \equiv \frac{2}{x + \sqrt{x^2 + 2}} \,.$$

Then we want to prove that $f(x) \leq R(x) \leq g(x)$. Now $f'(x) \geq xf(x) - 1$, $g'(x) \leq xg(x) - 1$, and $R'(x) = xR(x) - 1$ where all three functions are bounded above by $1/x$. Consideration of the function $(R - f)'$ shows that if $R - f$ is ever negative, it goes to $-\infty$, and similarly for $(g - R)'$.

Question: For what values of x is the upper bound in Inequality 2 better than the upper bound in Inequality 1? Similarly, for what values of x is the lower bound in Inequality 2 better than the lower bound in Inequality 1?