The Hewitt-Savage $0-1$ law

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Definition. Let $\mathbb{N} = \{1, 2, \ldots\}$; if $\pi : \mathbb{N} \to \mathbb{N}$ is a bijection such that $\pi(n) = n$ for all but finitely many n, then π is a *finite permutation* of $\mathbb{N}$. For $x \in R^\infty$, let $T_\pi(x) = x \circ \pi = (x_{\pi(1)}, x_{\pi(2)}, \ldots)$. Then $A \in \mathcal{B}^\infty$ is *symmetric* under finite permutations if $T_\pi^{-1}(A) = \{x \in R^\infty : T_\pi(x) \in A\} = A$. Let $\mathcal{S}$ be the collection of all symmetric sets $A \in \mathcal{B}^\infty$.

Theorem. (Hewitt-Savage $0-1$ law). Suppose that $\underline{X} = (X_1, X_2, \ldots)$ are i.i.d. random variables on $(\Omega, \mathcal{A}, P)$, and let $\mathcal{S} \subset \mathcal{B}^\infty$ be the symmetric sigma-field. If $A \in \underline{X}^{-1}(\mathcal{S})$, then $P(A) = 0$ or $P(A) = 1$.

Proof. Let Q denote the induced probability measure $P_{\underline{X}} = P \circ \underline{X} \equiv Q$ on $(R^\infty, \mathcal{B}^\infty)$. Let $\mathcal{F}_n = \mathcal{B}^n \times R^\infty$, and note that $\mathcal{S} \subset \mathcal{B}^\infty = \sigma[\cup_n \mathcal{B}^n]$. By exercise 1.2.2, for any $A \in \mathcal{S}$, there exist sets $B_n \in \mathcal{B}^n$ such that the corresponding sets $A_n = B_n \times R^\infty$ satisfy $Q(A \triangle A_n) \to 0$. Writing $\tilde{A}_n = R^n \times A_n \times R^\infty$, it is clear from the symmetry of Q and A that $Q(\tilde{A}_n) = Q(A_n) \to Q(A)$ and $Q(A \triangle \tilde{A}_n) = Q(A \triangle A_n) \to 0$. Hence

$$Q(A \triangle (A_n \cap \tilde{A}_n)) \le Q(A \triangle A_n) + Q(A \triangle \tilde{A}_n) \ \to 0\,.$$

Since A_n and $\tilde{A}_n$ are independent events, we also have

$$Q(A) \leftarrow Q(A_n \cap \tilde{A}_n) = Q(A_n) Q(\tilde{A}_n) \to Q(A) Q(A) = Q^2(A)\,.$$

Thus we have $Q^2(A) = Q(A)$, and this forces $Q(A) = 0$ or $Q(A) = 1$. □

Exercise 1. Show that $\mathcal{S}$ is a sigma-field.

Exercise 2. Show that $\mathcal{T} \subset \mathcal{S}$ where $\mathcal{T}$ is the tail $\sigma-$field.

Exercise 3. Give an example to show that $\mathcal{S} \not\subset \mathcal{T}$ where $\mathcal{T}$ is the tail $\sigma-$field. Hint: consider $[S_n \in A i.o.]$ for a Borel set A where $S_n = X_1 + \ldots + X_n$.