

Uniqueness Proof, Caratheodory extension theorem

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The goal here is provide additional details for the uniqueness proof in Pfs (2012), page 14.

Claim 7: When μ is a finite measure its extension μ^* to \mathcal{A}^* is unique.

Proof. Let ν denote any other extension of μ to \mathcal{A}^* , and let $A \in \mathcal{A}^*$. For any Caratheodory covering A_1, A_2, \dots of A with the A_n 's in \mathcal{C} , countable sub-additivity gives

$$\nu(A) \leq \nu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

since $\mu = \nu$ on \mathcal{C} . Thus by the definition of μ^* ,

$$\nu(A) \leq \mu^*(A). \quad (1)$$

Repeating this argument for A^c , let B_1, B_2, \dots be a Caratheodory covering of A^c . Then

$$\nu(A^c) \leq \nu(\cup_n B_n) \leq \sum_{n=1}^{\infty} \nu(B_n) = \sum_{n=1}^{\infty} \mu(B_n)$$

and hence

$$\nu(A^c) \leq \mu^*(A^c). \quad (2)$$

But we also have

$$\nu(A) + \nu(A^c) = \nu(\Omega) = \mu(\Omega) = \mu^*(A) + \mu^*(A^c). \quad (3)$$

Hence $\nu(A) = \mu^*(A)$ for all $A \in \mathcal{A}^*$. To see that this is implied by (1), (2), and (3), let $a \equiv \nu(A)$, $b \equiv \nu(A^c)$, $a^* \equiv \mu^*(A)$, and $b^* \equiv \mu^*(A^c)$. Then we have

$$a + b = a^* + b^*, \quad a \leq a^*, \quad b \leq b^* \quad \text{or} \quad (a, b) \leq (a^*, b^*).$$

These relations can be satisfied only if $a = a^*$ and $b = b^*$.