Borel and Lebesgue - Measurable Sets

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Suppose that $(\Omega, \mathcal{A}, \mu) = ([0,1], \mathcal{B}, \lambda)$. Two natural questions arose in section 1.2:

Question 1: Is every subset of $[0,1]$ Lebesgue - measurable?

Question 2: Is every Lebesgue measurable subset of $[0,1]$ a Borel-measurable set?

The answer to both questions is no: $\widehat{\mathcal{B}}_\lambda \subset 2^{[0,1]}$ and $\mathcal{B} \subset \widehat{\mathcal{B}}_\lambda$ with strict inclusion in both places. The proof of the first fact is given in PfS, Proposition 2.3, page 13; the second is proved in Cohn (1980). [The existence of a set A which is not Lebesgue-measurable is given in Cohn (1980), Theorem 1.4.7, page 32. The existence of a set which is Lebesgue-measurable but not Borel-measurable is given in Theorem 2.1.9, page 56.]

Theorem 1. (PfS page 13; Cohn, page 32). There exists a subset A of $(0,1)$ such that $A \notin \widehat{\mathcal{B}}_\lambda$.

Theorem 2. (Cohn, page 56). There exists a set $A \in \widehat{\mathcal{B}}_\lambda$ such that $A \notin \mathcal{B}$.

The proof of Theorem 2 involves the *Cantor function.* Let $K_0 \equiv [0,1]$. For each $n = 1, 2, \ldots$, let K_n be the compact set formed from K_{n-1} by removing from K_{n-1} the open middle third of each of the intervals making up K_{n-1}: $K_1 = [0,1] \setminus (1/3, 2/3)$, $K_2 = K_1 \setminus [(1/9, 2/9) \cup (7/9, 8/9)]$, ... Define the Cantor set K by $K_n \equiv \cap_{n=1}^\infty K_n$.

The Cantor singular distribution function F is the function $F : [0,1] \to [0,1]$ defined as follows: $F(x) = 1/2$ for $x \in (1/3, 2/3)$; $F(x) = 1/4$ for $x \in (1/9, 2/9)$ and $F(x) = 3/4$ for $x \in (7/9, 8/9)$; $\ldots$; $F(x) = 1/2^n, 3/2^n, 5/2^n, \ldots$ on the successive intervals removed from K_{n-1} in the construction of K_n. Thus F is defined on the open set $[0,1] \setminus K$, is nondecreasing, and has values in $[0,1]$. Extend it to all of $[0,1]$ by letting $F(0) = 0$, and setting

$$F(x) \equiv \sup\{F(y) : t \in [0,1] \setminus K \quad \text{and} \quad y < x\}$$

for $x \in K$ and $x \neq 0$.

It is not hard to check that F is non-decreasing and continuous with $F(0) = 0$ and $F(1) = 1$. Because F is continuous, its range is all of $[0,1]$. Now the inverse (or quantile) function F^{-1} of F defined by

$$F^{-1}(y) \equiv \inf\{x \in [0,1] : \ F(x)] \geq y\}$$

is one-to-one (injective) and $F^{-1}([0,1]) \subset K$. Because F^{-1} is non-decreasing it is Borel-measurable.

Proof of Theorem 2. With the above preparation, the proof of Theorem 2 follows essentially from Theorem 1. Let $A \subset [0,1]$ be a set which is not Lebesgue - measurable; such a set exists by Theorem 1. Let $B = F^{-1}(A)$. Then B is a subset of the Cantor set, and hence is Lebesgue-measurable: by the construction of K it follows that $\lambda(K) = 0$ and by the definition of $\widehat{\mathcal{B}}_\lambda$ this forces $B \in \widehat{\mathcal{B}}_\lambda$. If B were a Borel set, then $(F^{-1})^{-1}(B)$ would also be a Borel set since F^{-1} is Borel - measurable. However the one-to-one-ness of F^{-1} implies that $(F^{-1})^{-1}(B) = A$ which is not Lebesgue-measurable, and hence is not a Borel set. Thus we conclude that B is Lebesgue-measurable but not a Borel set. $\square$