CHAPTER 1. MEASURES: SUPPLEMENT

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1. Completion and μ^*−measurable sigma-fields

Theorem 1.1 (Minimal Measurable Cover theorem)**.** *(Dudley, RAP (2002), page 101). For any measure space $(\Omega, \mathcal{A}, \mu)$ and any $B \subset \Omega$ there is a set $C \in \mathcal{A}$ with $B \subset C$ and $\mu^*(B) = \mu(C)$.*

Remark 1.2. *In the notation and terminology of van der Vaart and Wellner (1996), $C = B^*$ and we have $\mu^*(B) = \mu(B^*)$. Such a set $C = B^*$ is called the* minimal measurable cover *of B. See van der Vaart and Wellner (1996), section 1.2, pages 6-9 and especially Lemma 1.2.3 on page 9. This goes back to Blumberg (1935) and Eames and May (1967).*

Proof. If $B \subset \cup_n A_n$ and $A_n \in \mathcal{A}$, let $A \equiv \cup_n A_n$. Then $B \subset A$ and $A \in \mathcal{A}$. By countable sub-additivity of μ, $\mu(A) \le \sum_{n=1}^{\infty} \mu(A_n)$. Hence $\mu^*(B) = \inf\{\mu(A) : \ B \subset A, \ A \in \mathcal{A}\}$. For each $m = 1, 2, \ldots$ choose $C_m \in \mathcal{A}$ with $B \subset C_m$ and $\mu(C_m) \le \mu^*(B) + 1/m$. Let $C \equiv \cap_m C_m$. Then $C \in \mathcal{A}$ since $\mathcal{A}$ is closed under countable intersections, $B \subset C$, and $\mu^*(B) = \mu(C)$ since $\mu(C) = \lim_m \mu(C_m)$ by Proposition 1.2(b). □

Here is a preliminary result of Dudley's connecting two ways of completing $(\Omega, \mathcal{A}, \mu)$. Let $\mathcal{N}(\mu) \equiv \{F \subset \Omega : \ \mu^*(F) = 0\}$. Note that if $G \subset \cup_n F_n$ with $F_n \in \mathcal{N}(\mu)$, then $G \in \mathcal{N}(\mu)$. Let $\mathcal{A} \vee \mathcal{N}(\mu) \equiv \sigma[\mathcal{A} \cup \mathcal{N}(\mu)]$, and define

$$\begin{aligned} \mathcal{A}_\mu^* &= \{A \subset \Omega : \ A \Delta B \in \mathcal{N}(\mu) \text{ for some } B \in \mathcal{A}\} \\ &= \{\text{all almost equal to } \ B \text{ sets } \ A \ \text{ for some } B \in \mathcal{A}\}. \end{aligned}$$

Proposition 1.3. *(Dudley, RAP (2002), pages 101-102). For any measure space $(\Omega, \mathcal{A}, \mu)$ and $\mathcal{A}_\mu^*$ as defined above, $\mathcal{A}_\mu^* = \mathcal{A} \vee \mathcal{N}(\mu)$.*

Proof. If $A\Delta B \in \mathcal{N}(\mu)$ and $B \in \mathcal{A}$, then $A \setminus B \in \mathcal{N}(\mu)$ and $B \setminus A \in \mathcal{N}(\mu)$, so $A = (B \cup (AB^c)) \setminus (B \setminus A) \in \mathcal{A} \vee \mathcal{N}(\mu)$. Thus $\mathcal{A}_\mu^* \subset \mathcal{A} \cup \mathcal{N}(\mu)$.

To prove the reverse inclusion, first note that $\mathcal{A} \subset \mathcal{A}_\mu^*$ and $\mathcal{N}(\mu) \subset \mathcal{A}_\mu^*$. Furthermore, if $A_n \Delta B_n \in \mathcal{N}(\mu)$ for all n, then

$$(\cup_n A_n)\Delta(\cup_n B_n) \subset \bigcup_n (A_n \Delta B_n) \in \mathcal{N}(\mu)$$

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and $(\Omega \setminus A_1)\Delta(\Omega \setminus B_1) = A_1 \Delta B_1 \in \mathcal{N}(\mu)$. Thus $\mathcal{A}_\mu^*$ is a $\sigma-$field containing both $\mathcal{A}$ and $\mathcal{N}(\mu)$, and hence

$$\mathcal{A} \vee \mathcal{N}(\mu) \subset \mathcal{A}_\mu^*.$$

Combining the two inclusions completes the proof of the claimed equality. □

Here is a proposition connecting these developments in Dudley with the definitions of $\widehat{\mathcal{A}}_\mu$ given in Shorack, *Probability for Statisticians*, page 15.

Proposition 1.4. *$\widehat{\mathcal{A}}_\mu = \mathcal{A}_\mu^* = \mathcal{A} \vee \mathcal{N}(\mu)$.*

Proof. The second equality has been proved already in Proposition **??**.

Let $A \in \widehat{\mathcal{A}}_\mu$. Then (from the first definition in Exercise 2.1 of Shorack), there exist $A_1, A_2 \in \mathcal{A}$ with $A_1 \subset A \subset A_2$ and $\mu(A_2 \setminus A_1) = 0$. Now take $B = A_1$ in the definition of $\mathcal{A}_\mu^*$: with $B = A_1$, $A\Delta A_1 = AA_1^c$ where $\mu^*(AA_1^c) \le \mu^*(A_2 A_1^c) = \mu(A_2 \setminus A_1) = 0$. Thus $A \in \mathcal{A}_\mu^*$. (Similarly, if we take $B = A_2$ in the definition of $\mathcal{A}_\mu^*$, then with $B = A_2$, $A\Delta A_2 = A^c A_2$ where $\mu^*(A^c A_2) \le \mu^*(A_2 \cap A_1^c) = \mu(A_2 \setminus A_1) = 0$, so again $A \in \mathcal{A}_\mu^*$.) Thus $\widehat{\mathcal{A}}_\mu \subset \mathcal{A}_\mu^*$.

To show the reverse inclusion, it is convenient to use $\mathcal{A}_\mu^* = \mathcal{A} \vee \mathcal{N}(\mu)$, and the second description of $\widehat{\mathcal{A}}_\mu$ in Shorack's exercise 1.2.1:

$$\widehat{\mathcal{A}}_\mu = \{A \cup N : \ A \in \mathcal{A}, \text{ and } N \subset (\text{some } B) \in \mathcal{A} \text{ having } \mu(B) = 0\}.$$

Thus we clearly have $\widehat{\mathcal{A}}_\mu \subset \mathcal{A} \vee \mathcal{N}(\mu)$, which also follows from what has been proved above.

On the other hand, $\mathcal{A} \subset \widehat{\mathcal{A}}_\mu$ and

$$\{N \subset \Omega : \ \mu^*(N) = 0\} \subset \{N \subset \Omega : \ N \subset (\text{some } B \in \mathcal{A} \text{ with } \mu(B) = 0\}$$

by taking $B = N^*$ from Theorem **??** with $N \subset N^*$ and $\mu(N^*) = \mu^*(N) = 0$. Thus

$$\mathcal{A} \vee \mathcal{N}(\mu) = \sigma[\mathcal{A} \cup \mathcal{N}(\mu)] \subset \widehat{\mathcal{A}}_\mu.$$

Combining the two inclusions yields the claim. □

Theorem 1.5 (Completion equals μ^*-measurable for sigma-finite μ)**.** *(Dudley, RAP (2002), page 103). For any $\sigma-$finite measure space $(\Omega, \mathcal{A}, \mu)$, $\widehat{\mathcal{A}}_\mu = \mathcal{A}^*$ (or, in the notation of RAP, $\mathcal{A} \vee \mathcal{N}(\mu) = \mathcal{M}(\mu^*)$).*

Proof. Let $\Omega = \sum_n \Omega_n$ with $\mu(\Omega_n) < \infty$ for each $n \ge 1$ and let $A \in \mathcal{A}^*$. To show that $A \in \widehat{\mathcal{A}}_\mu$ it suffices to show that $A \cap \Omega_n \in \widehat{\mathcal{A}}_\mu$ for all n. Thus we can assume that μ is finite. Let $B \in \mathcal{A}$ be a minimal measurable cover for A by Theorem **??**. Thus $A^* = B \supset A$ and $\mu^*(A) = \mu(B)$. Then $\mu^*(A) = \mu(B) = \mu^*(B) = \mu^*(BA) + \mu^*(BA^c) = \mu^*(A) + \mu^*(BA^c)$. This implies that $\mu^*(BA^c) = 0$ and $A = B \setminus (B \setminus A) \in \widehat{\mathcal{A}}_\mu$. Thus we have shown that $\mathcal{A}^* \subset \widehat{\mathcal{A}}_\mu$. The reverse inclusion, $\widehat{\mathcal{A}}_\mu \subset \mathcal{A}^*$, has already been proved in our proof of the Caratheodory extension theorem together with the fact that $\mu^*(N) = 0$ implies that $N \in \mathcal{A}^*$; so the claimed equality, $\mathcal{A}_\mu^* = \mathcal{A}^*$, follows. □

Remark 1.6. *Theorem **??** is stated in Cohn, Measure Theory, page 42; see Problem 9.*