then $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = \mu(R^1)$. If μ is a probability measure, F is called a *distribution function* (the adjective *cumulative* is sometimes added).

Measures μ are often specified by means of the function F. The following theorem ensures that to each F there does exist a μ.

Theorem 12.4. *If F is a nondecreasing, right-continuous real function on the line, there exists on $\mathscr{R}^1$ a unique measure μ satisfying* (12.5) *for all a and b.*

As noted above, uniqueness is a simple consequence of Theorem 10.3. The proof of existence is almost the same as the construction of Lebesgue measure, the case $F(x) = x$. This proof is not carried through at this point, because it is contained in a parallel, more general construction for k-dimensional space in the next theorem. For a very simple argument establishing Theorem 12.4, see the second proof of Theorem 14.1.

Specifying Measures in R^k

The σ-field $\mathscr{R}^k$ of k-dimensional Borel sets is generated by the class of bounded rectangles

$$A = [x: a_i < x_i \le b_i,\ i = 1, \dots, k] \tag{12.7}$$

(Example 10.1). If $I_i = (a_i, b_i]$, A has the form of a cartesian product

$$A = I_1 \times \cdots \times I_k. \tag{12.8}$$

Consider the sets of the special form

$$S_x = [y: y_i \le x_i,\ i = 1, \dots, k]; \tag{12.9}$$

S_x consists of the points "southwest" of $x = (x_1, \dots, x_k)$; in the case $k = 1$ it is the half-infinite interval $(-\infty, x]$. Now S_x is closed, and (12.7) has the form

$$A = S_{(b_1 \cdots b_k)} - \left[S_{(a_1 b_2 \cdots b_k)} \cup S_{(b_1 a_2 \cdots b_k)} \cup \cdots \cup S_{(b_1 b_2 \dots a_k)} \right]. \tag{12.10}$$

Therefore, the class of sets (12.9) generates $\mathscr{R}^k$. This class is a π-system.

The objective is to find a version of Theorem 12.4 for k-space. This will in particular give k-dimensional Lebesgue measure. The first problem is to find the analogue of (12.5).

A bounded rectangle (12.7) has 2^k vertices—the points $x = (x_1, \dots, x_k)$ for which each x_i is either a_i or b_i. Let $\operatorname{sgn}_A x$, the signum of the vertex, be

$+1$ or -1, according as the number of i $(1 \le i \le k)$ satisfying $x_i = a_i$ is even or odd. For a real function F on R^k, the difference of F around the vertices of A is $\Delta_A F = \sum \operatorname{sgn}_A x \cdot F(x)$, the sum extending over the 2^k vertices x of A. In the case $k = 1$, $A = (a, b]$ and $\Delta_A F = F(b) - F(a)$. In the case $k = 2$, $\Delta_A F = F(b_1, b_2) - F(b_1, a_2) - F(a_1, b_2) + F(a_1, a_2)$.

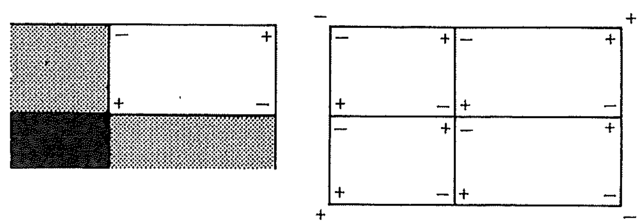
Since the k-dimensional analogue of (12.4) is complicated, suppose at first that μ is a finite measure on $\mathscr{R}^k$ and consider instead the analogue of (12.6), namely

(12.11) $$F(x) = \mu[y\colon y_i \le x_i,\ i = 1, \ldots, k].$$

Suppose that S_x is defined by (12.9) and A is a bounded rectangle (12.7). Then

(12.12) $$\mu(A) = \Delta_A F.$$

To see this, apply to the union on the right in (12.10) the inclusion-exclusion formula (10.5). The k sets in the union give $2^k - 1$ intersections, and these are the sets S_x for x ranging over the vertices of A other than $(b_1, \ldots, b_k)$. Taking into account the signs in (10.5) leads to (12.12).



Suppose $x^{(n)} \downarrow x$ in the sense that $x_i^{(n)} \downarrow x_i$ as $n \to \infty$ for each $i = 1, \ldots, k$. Then $S_{x^{(n)}} \downarrow S_x$, and hence $F(x^{(n)}) \to F(x)$ by Theorem 10.2(ii). In this sense, F is *continuous from above*.

Theorem 12.5. *Suppose that the real function F on R^k is continuous from above and satisfies $\Delta_A F \ge 0$ for bounded rectangles A. Then there exists a unique measure μ on $\mathscr{R}^k$ satisfying* (12.12) *for bounded rectangles A.*

The empty set can be taken as a bounded rectangle (12.7) for which $a_i = b_i$ for some i, and for such a set A, $\Delta_A F = 0$. Thus (12.12) defines a finite-valued set function μ on the class of bounded rectangles. The point of the theorem is that μ extends uniquely to a measure on $\mathscr{R}^k$. The uniqueness is an immediate consequence of Theorem 10.3, since the bounded rectangles form a π-system generating $\mathscr{R}^k$.

If F is bounded, then μ will be a finite measure. But the theorem does not require that F be bounded. The most important unbounded F is $F(x) = x_1 \cdots x_k$. Here $\Delta_A F = (b_1 - a_1) \cdots (b_k - a_k)$ for A given by (12.7). This is the ordinary volume of A as specified by (12.1). The corresponding measure extended to $\mathscr{R}^k$ is k-dimensional Lebesgue measure as described at the beginning of this section.

PROOF OF THEOREM 12.5. As already observed, the uniqueness of the extension is easy to prove. To prove its existence it will first be shown that μ as defined by (12.12) is finitely additive on the class of bounded rectangles. Suppose that each side $I_i = (a_i, b_i]$ of a bounded rectangle (12.7) is partitioned into n_i subintervals $J_{ij} = (t_{i,j-1}, t_{ij}]$, $j = 1, \ldots, n_i$, where $a_i = t_{i0} < t_{i1} < \cdots < t_{in_i} = b_i$. The $n_1 n_2 \cdots n_k$ rectangles

$$B_{j_1 \cdots j_k} = J_{1j_1} \times \cdots \times J_{kj_k}, \qquad 1 \le j_1 \le n_1, \ldots, 1 \le j_k \le n_k, \tag{12.13}$$

then partition A. Call such a partition *regular*. It will first be shown that μ adds for regular partitions:

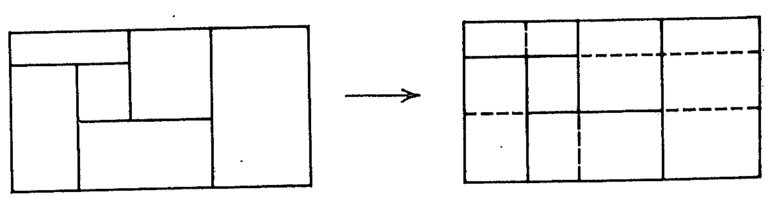
$$\mu(A) = \sum_{j_1 \cdots j_k} \mu\left(B_{j_1 \cdots j_k}\right). \tag{12.14}$$

The right side of (12.14) is $\Sigma_B \Sigma_x \operatorname{sgn}_B x \cdot F(x)$, where the outer sum extends over the rectangles B of the form (12.13) and the inner sum extends over the vertices x of B. Now

$$\sum_B \sum_x \operatorname{sgn}_B x \cdot F(x) = \sum_x F(x) \sum_B \operatorname{sgn}_B x, \tag{12.15}$$

where on the right the outer sum extends over each x that is a vertex of one or more of the B's, and for fixed x the inner sum extends over the B's of which it is a vertex. Suppose that x is a vertex of one or more of the B's but is not a vertex of A. Then there must be an i such that x_i is neither a_i nor b_i. There may be several such i, but fix on one of them and suppose for notational convenience that it is $i = 1$. Then $x_1 = t_{1j}$ with $0 < j < n_1$. The rectangles (12.13) of which x is a vertex therefore come in pairs $B' = B_{jj_2 \cdots j_k}$ and $B'' = B_{j+1, j_2 \cdots j_k}$, and $\operatorname{sgn}_{B'} x = -\operatorname{sgn}_{B''} x$. Thus the inner sum on the right in (12.15) is 0 if x is not a vertex of A.

On the other hand, if x is a vertex of A as well as of at least one B, then for each i either $x_i = a_i = t_{i0}$ or $x_i = b_i = t_{in_i}$. In this case x is a vertex of only one B of the form (12.13)—the one for which $j_i = 1$ or $j_i = n_i$, according as $x_i = a_i$ or $x_i = b_i$—and $\operatorname{sgn}_B x = \operatorname{sgn}_A x$. Thus the right side of (12.15) reduces to $\Delta_A F$, which proves (12.14).



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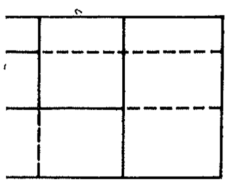

Now suppose that $A = \bigcup_{u=1}^n A_u$, where A is the bounded rectangle (12.8), $A_u = I_{1u} \times \cdots \times I_{ku}$ for $u = 1, \ldots, n$, and the A_u are disjoint. For each i ($1 \le i \le k$), the intervals $I_{i1}, \ldots, I_{in}$ have I_i as their union, although they need not be disjoint. But their endpoints split I_i into disjoint subintervals $J_{i1}, \ldots, J_{in_i}$ such that each I_{iu} is the union of certain of the J_{ij}. The rectangles B of the form (12.13) are a regular partition of A as before; furthermore, the B's contained in a single A_u form a regular partition of A_u. Since the A_u are disjoint, it follows by (12.14) that

$$\mu(A) = \sum_B \mu(B) = \sum_{u=1}^n \sum_{B \subset A_u} \mu(B) = \sum_{u=1}^n \mu(A_u).$$

Therefore, μ is finitely additive on the class $\mathscr{I}^k$ of bounded k-dimensional rectangles.

As shown in Example 11.5, $\mathscr{I}^k$ is a semiring, and so Theorem 11.3 applies.

Now suppose that $\bigcup_{u=1}^n A_u \subset A$, where A and the A_u are in $\mathscr{I}^k$ and the latter are disjoint. As shown in Example 11.2, $A - \bigcup_{u=1}^n A_u$ is a finite disjoint union $\bigcup_{v=1}^m B_v$ of sets in $\mathscr{I}^k$. Therefore, $\sum_{u=1}^n \mu(A_j) + \sum_{v=1}^m \mu(B_v) = \mu(A)$, and so

$$\text{(12.16)} \qquad \sum_{u=1}^n \mu(A_u) \le \mu(A) \quad \text{if} \quad \bigcup_{u=1}^n A_u \subset A, \qquad A_u \cap A_v = 0.$$

Now suppose that $A \subset \bigcup_{u=1}^n A_u$, where A and the A_u are in $\mathscr{I}^k$. Let $B_1 = A \cap A_1$ and $B_u = A \cap A_u \cap A_1^c \cap \cdots \cap A_{u-1}^c$. Again, each B_u is a finite disjoint union $\bigcup_v B_{uv}$ of elements of $\mathscr{I}^k$. The B_u are disjoint, and so the B_{uv} taken all together are also disjoint. They have union A, so that $\mu(A) = \sum_u \sum_v \mu(B_{uv})$. Since $\bigcup_v B_{uv} = B_u \subset A_u$, an application of (12.16) gives

$$\text{(12.17)} \qquad \mu(A) \le \sum_{u=1}^n \mu(A_u) \quad \text{if} \quad A \subset \bigcup_{u=1}^n A_u.$$

Of course, (12.16) and (12.17) give back finite additivity again.

To apply Theorem 11.3 requires showing that μ is countably subadditive on $\mathscr{I}^k$. Suppose then that $A \subset \bigcup_{u=1}^\infty A_u$, where A and the A_u are in $\mathscr{I}^k$. The problem is to prove that

$$\text{(12.18)} \qquad \mu(A) \le \sum_{u=1}^\infty \mu(A_u).$$

Suppose that $\epsilon > 0$. If A is given by (12.7) and $B = [x\colon a_i + \delta < x_i \le b_i, i \le k]$, then $\mu(B) > \mu(A) - \epsilon$ for small enough positive δ because μ is defined by (12.12) and F is continuous from above. Note that A contains the closure $B^- = [x\colon a_i + \delta \le x_i \le b_i, i \le k]$ of B. Similarly, for each u there is in $\mathscr{I}^k$ a set $B_u = [x\colon a_{iu} < x_i \le b_{iu} + \delta_u, i \le k]$ such that $\mu(B_u) < \mu(A_u) + \epsilon/2^u$ and A_u is in the interior $B_u^\circ = [x\colon a_{iu} < x_i < b_{iu} + \delta_u, i \le k]$ of B_u.

Since $B^- \subset A \subset \bigcup_{u=1}^\infty A_u \subset \bigcup_{u=1}^\infty B_u^\circ$, it follows by the Heine–Borel theorem that $B \subset B^- \subset \bigcup_{u=1}^n B_u^\circ \subset \bigcup_{u=1}^n B_u$ for some n. Now (12.17) applies, and so $\mu(A) - \epsilon < \mu(B) \le \sum_{u=1}^n \mu(B_u) < \sum_{u=1}^\infty \mu(A_u) + \epsilon$. Since ϵ was arbitrary, the proof of (12.18) is complete.