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PRODUCT MEASURES AND EXTENSION THEOREMS

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1. Kolmogorov's extension theorem. Here we restate Definition 5.3.1 and Theorem 5.3.1 from PfS, page 95:

Definition 5.3.1 (Consistency) Finite-dimensional distributions $\{(\mathbb{R}^n, \mathcal{B}_n, P_n)\}_{n=1}^{\infty}$ are *consistent* if for every $n \ge 1$, every $B_1, \ldots, B_n \in \mathcal{B}$, and every $1 \le i \le n$,

$$
\begin{aligned}
&P_{n-1}((X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \in B_1 \times \cdots \times B_{i-1} \times B_{i+1} \times \cdots \times B_n) \\
&\quad = \quad P_n((X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_n) \in B_1 \times \cdots \times B_{i-1} \times \mathbb{R} \times B_{i+1} \times \cdots \times B_n).
\end{aligned}
$$

Theorem 5.3.1 (Kolmogorov's extension theorem) An extension of any consistent family of probability measures $\{(\mathbb{R}^n, \mathcal{B}_n, P_n)\}_{n=1}^{\infty}$ to a probability $P(\cdot)$ on $(R^{\infty}, \mathcal{B}_{\infty})$ necessarily exists, and it is unique.

Theorem 5.3.2 ($(\mathbb{R}^{\infty}, \mathcal{B}_{\infty})$-extension theorem; Breiman) Let P on $\mathcal{C}_I$ satisfy:
(a) $P \ge 0$ and $P(\mathbb{R}^{\infty}) = 1$.
(b) If $D = \sum_{j=1}^{m} D_j$ for $n-$dimensional rectangles D and D_j, then $P(D) = \sum_{j=1}^{m} P(D_j)$.
(c) If D denotes any fixed $n-$dimensional rectangle, then there exists a sequence of compact $n-$dimensional rectangles D_j for which $D_j \nearrow D$ and $P(D_j) \nearrow P(D)$. [That is, P is well-defined and additive on $n-$dimensional rectangles and satisfies something like continuity from below.] Then there exists a unique extension of P to $\mathcal{B}_{\infty}$.

These results rely upon the topological properties of $\mathbb{R}$ crucially. They do imply the following important corollary.

Corollary. (Product measures on $(\mathbb{R}^{\infty}, \mathcal{B}_{\infty})$). Suppose $Q_1, Q_2, \ldots$ are probability measures on $\mathbb{R}$ and P_n is defined on $(\mathbb{R}^n, \mathcal{B}_n)$ for each $n \ge 1$ by

$$P_n((X_1, \ldots, X_n) \in B_1 \times \cdots \times B_n) = Q_1(B_1) \cdots Q_n(B_n)$$

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for all Borel sets $B_j \in \mathcal{B}$, $1 \le j \le n$. Then the P_n's form a consistent collection of finite-dimensional distributions and hence there is a product measure $P = \prod_{j=1}^{\infty} Q_j$ on $(\mathbb{R}^{\infty}, \mathcal{B}_{\infty})$ corresponding to the Q_j's.

Question. What if the Q_j's in the corollary are associated with arbitrary measurable spaces $(\Omega_j, \mathcal{A}_j)$ for $j \ge 1$ rather than $(\mathbb{R}, \mathcal{B})$? Does the product measure $P = \prod_{j=1}^{\infty} Q_j$ exist as a measure on $(\Omega, \mathcal{A})$ where $\Omega = \Omega_1 \times \Omega_2 \times \cdots$ and $\mathcal{A}$ is the product sigma-field? The next section gives an affirmative answer to this question without imposing any topological assumptions on the spaces Ω_j.

As noted by Dudley (2002), page 256 and problem 2, section 12.1, page 448, existence of consistent marginal distributions alone does not imply existence of a probability measure on the product measurable space in general. Apparently the first counterexample was given by Andersen and Jessen (1948a).

2. Existence of infinite product probability measures without topology. Here we get product probability measures without topological hypotheses. The results below are taken from Dudley (2002), section 8.2, pages 255-259, but apparently are originally due to Lomnicki and Ulam (1934), von Neumann's lectures from (1935) published in von Neumann (1950), Jessen (1939) (in Danish), Kakutani (1943a), Andersen and Jessen (1946), Andersen and Jessen (1948a), Andersen and Jessen (1948b). See Dudley (2002) page 275 for more on the history of these results.

Let $(\Omega_n, \mathcal{A}_n, P_n)$ be probability spaces for each n. Let Ω be the Cartesian product $\Omega = \prod_{j=1}^{\infty} \Omega_j$; i.e

$$\Omega = \{(\omega_1, \omega_2, \ldots) : \ \omega_j \in \Omega_j \ \text{ for all } j \ge 1\}.$$

Let $\mathcal{A}$ be the smallest $\sigma-$field of subsets of Ω such that $\pi_n(\{\omega_m\}_{m \ge 1}) = \omega_n$ (the projection map from Ω to Ω_n) is measurable for every n. Thus

$$\mathcal{A} = \left\{ \begin{array}{l} \text{smallest } \sigma-\text{field containing all subsets } \ \pi_n^{-1}(A) \\ \text{for all } \ n \text{ and all } \ A \in \mathcal{A}_n \end{array} \right\}.$$

Let

$$\begin{aligned} \mathcal{R} \ &= \ \left\{ \begin{array}{l} \prod_n A_n \subset \otimes : \ A_n \in \mathcal{A}_n \text{ for all } n \\ \text{and } A_n = \Omega_n \ \text{ except for at most finitely many values of } n \end{array} \right\} \\ &= \ \{\text{all finite-dimensional rectangles}\}. \end{aligned}$$

Proposition 1. The collection $\mathcal{R}$ all finite-dimensional rectangles in the infinite product Ω is a $\overline{\pi}-$system and a semiring: $\emptyset \in \mathcal{R}$, and if $A, B \in \mathcal{R}$, the $A \cap B \in \mathcal{R}$, and $A \setminus B = \cup_{j=1}^{m} C_j$ where $C_j \in \mathcal{R}$ and n is finite. The field $\mathcal{C}_F$ generated by $\mathcal{R}$ is the collection of finite disjoint unions of elements of $\mathcal{R}$.

Proof. Note that $\emptyset \in \mathcal{R}$ since $\emptyset \in \mathcal{A}_n$ for each n. Also, $\Omega \in \mathcal{R}$ since $\Omega_n \in \mathcal{A}_n$ for each n. If $C, D \in \mathcal{R}$, then $C \cap D$ is a finite-dimensional rectangle (i.e $C \cap D \in \mathcal{R}$). Thus $\mathcal{R}$ is a $\overline{\pi}$ system. For a product of two spaces, the collection of rectangles is a semi-ring: specifically, the difference of two rectangles is a union of two disjoint rectangles:

$$(A \times B) \setminus (E \times F) = (A \setminus E) \times B) \cup ((A \cap E) \times (B \setminus F)).$$

If follows by induction that in any finite Cartesian product, any difference $C \setminus D$ of two rectangles is a finite disjoint union of rectangles. Thus $\mathcal{R}$ is a semi-ring. We have $\Omega \in \mathcal{R}$, so the ring generated by $\mathcal{R}$ is a field. $\square$

Now for $A = \prod_n A_n \in \mathcal{R}$, let $P(A) = \prod_n P_n(A_n)$. The product converges since all but finitely many factors are 1.

Theorem 1. (Existence theorem for infinite product probability spaces) P on $\mathcal{R}$ extends uniquely to a countably additive probability measure on $\mathcal{A}$.

Proof. Let $\mathcal{C}_F$ denote the field of subsets of Ω generated by $\mathcal{R}$. For each $A \in \mathcal{C}_F$, write $A = \sum_{j=1}^{m} A_j$ with $A_j \in \mathcal{R}$, and define $P(A) = \sum_{j=1}^{m} P(A_j)$.
• We first show that P is well-defined and additive on $\mathcal{C}_F$.
• We then show that P is countably additive on $\mathcal{C}_F$. Then the Caratheodory extension theorem applies to give the conclusion.

So, first: P is well-defined and additive on $\mathcal{C}_F$. for $A = \sum_{j=1}^{N}$ with $A_j \in \mathcal{R}$, each A_j is a product of sets $B_{jn} \in \mathcal{A}_n$ with $B_{jn} = \Omega_n$ for all $n \geq n(j)$ for some $n(j) < \infty$. Let $m = \max_{1 \leq j \leq N} n(j)$. Then, since all the $B_{jn} = \Omega_n$ for $n > m$, properties of P on such sets are equivalent to properties of the finite product measure on $\Omega_1 \times \cdots \times \Omega_m$. To show that P is well-defined, if $A = \sum_{j=1}^{N} A_j = \sum_{j=1}^{N'} A'_j$, taking $m = m_A \vee m_{A'}$ still yields a finite product. So P is well-defined and finitely additive on $\mathcal{A}$ by the finite product-measure theorem.

Now we need to show that P is countably additive on $\mathcal{C}_F$. To do this, it suffices by Theorem 3.1.1 Dudley (2002) (or Proposition 1.1.4 PfS), to show that if $A_j \in \mathcal{C}_F$ satisfy $A_1 \supset A_2 \supset \cdots$ and $\cap_{j=1}^{\infty} A_j \ \emptyset$ (or $\lim_j A_j = \emptyset$), then $P(A_j) \searrow 0$. Thus it suffices to show that if $A_j \searrow$ and $\liminf_j P(A_j) \geq \epsilon$, then $\cap_{j=1}^{\infty} A_j \neq \emptyset$.

Let $P^{(0)} \equiv P$ on $\mathcal{C}_F$. For each $n \geq 1$, let $\Omega^{(n)} \equiv \prod_{m>n} \Omega_m$. Then let $\mathcal{C}_F^{(n)}$ and $P^{(n)}$ be defined on $\Omega^{(n)}$ just as $\mathcal{C}_F$ and P were defined on Ω.
For each set $E \subset \Omega$ and $x_i \in \Omega_i$, $i = 1, \ldots, n$, let

$$E^{(n)}(x_1, \ldots, x_n) = \{\{x_m\}_{m>n} \in \Omega^{(n)} : \ \underline{x} = \{x_i\}_{i \geq 1} \in E\}.$$

For a set $A \in \mathcal{X} \times \mathcal{Y}$ and $x \in \mathcal{X}$, let

$$A_x = \{y \in \mathcal{Y} : \ (x, y) \in A\} = \text{the } x - \text{section of } A.$$

Note that if $A \in \mathcal{S} \times \mathcal{T}$, a sigma-field for $\mathcal{X} \times \mathcal{Y}$, then $A_x \in \mathcal{T}$. For and $E \in \mathcal{C}_F$ there is an N large enough so that $E = F \times \prod_{n>N} \Omega_n$ for some $F \subset \prod_{n \leq N} \Omega_n$. [Since E is a finite union of rectangles with this property, take the maximum of the values of N for the rectangles.] Then $F = \sum_{k=1}^m F_k$ where $F_k = \prod_{i=1}^N F_{ki}$ some $F_{k,i} \in \mathcal{A}_i$, for $i = 1, \ldots, N$ and $k = 1, \ldots, m$. Now for any $n < N$ and $x_i \in \Omega_i$ for $i = 1, \ldots, n$,

$$E^{(n)}(x_1, \ldots, x_n) = G \times \Omega^{(N)}$$

where G is the union of those sets $\prod_{n<i\leq N} F_{k,i}$ such that $x_i \in F_{k,i}$ for all $1 \leq i \leq n$. Hence $E^{(n)}(x_1, \ldots, x_n) \in \mathcal{C}_F^{(n)}$ so $P^{(n)}(E^{(n)}(x_1, \ldots, x_n))$ is defined. By the Tonelli-Fubini theorem in $\Omega_1 \times \cdots \times \Omega_n \times \prod_{n<i\leq N} \Omega_i$ we have

$$P(E) = \int P^{(n)}(E^{(n)}(x_1, \ldots, x_n)) \prod_{1 \leq j \leq n} dP_j(x_j). \tag{1}$$

Now for the $\epsilon > 0$ with $P(A_j) \geq \epsilon > 0$ for all j, let

$$F_j \equiv \{x_1 \in \Omega_1 : \ P^{(1)}(A_j^{(1)}(x_1)) > \epsilon/2\}.$$

for each j, apply (1) to $E = A_j$. For $n = 1$ this yields

$$\begin{aligned}
\epsilon \leq P(A_j) &= \int P^{(1)}(A_j^{(1)}(x_1)) dP_1(x_1) \\
&= \left(\int_{F_j} + \int_{F_j^c} \right) P^{(1)}(A_j^{(1)}(x_1)) dP_1(x_1) \\
&\leq P_1(F_j) + \epsilon/2.
\end{aligned}$$

Thus $P_1(F_j) \geq \epsilon/2$ for all j where $A_j \searrow$ implies $A_j^{(1)} \searrow$ and $F_j \searrow$. Since P_1 is countably additive,

$$P_1(\cap_{j=1}^\infty F_j) \geq \epsilon/2 \quad \text{by the monotone convergence theorem,}$$

so $\cap_{j=1}^{\infty} F_j \neq \emptyset$. Take any $y_1 \in \cap_j F_n$. Let

$$\begin{aligned}
&f_j(y,x) \equiv P^{(2)}(A_j^{(2)}(y,x)),\\
&G_j \equiv \{x_2 \in \Omega_2 : \ f_j(y_1,x_2) > \epsilon/4\}.
\end{aligned}$$

Then $G_j \searrow$ as $j \to \infty$, so $\cap_j G_j \neq \emptyset$ in Ω_2, and we can choose $y_2 \in \cap_j G_j$.

Continuing inductively, the same argument yields $y_n \in \Omega_n$ for all n such that $P^{(n)}(A_j^{(n)}(y_1, y_2, \ldots, y_n)) \geq \epsilon/2^n$ for all j and n. Let $y \equiv \{y_n\}_{n\geq 1} \in \Omega$. To prove that $y \in A_j$ for each j, choose n large enough (depending on j) so that for all $x_1, \ldots, x_n$, $A_j^{(n)}(x_1, \ldots, x_n) = \emptyset$ or $\Omega^{(n)}$. This can be done since $A_j \in \mathcal{C}_F$. Then $A_j^{(n)}(y_1, \ldots, y_n) = \Omega^{(n)}$, and hence $y \in A_j$. Thus $\cap_{j\geq 1} A_j \neq \emptyset$. $\square$

As noted by Dudley (2002) page 259, this argument goes through for arbitrary (not necessarily countable) products of probability spaces. The resulting theorem has often been called the *Andersen-Jessen theorem*. See, e.g. Loève (1977), page 92. But other recent textbooks, in agreement with Dudley (2002), assign this theorem to Lomnicki and Ulam (1934); see e.g. Kallenberg (1997), page 93.

Going beyond product measures, there are further extension theorems which avoid topological hypotheses due to Ionescu Tulcea (1949); these theorems are treated and given further discussion by Kallenberg (1997) and Pollard (2002).

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