## Statistics 394, Problem Set 7 Solutions Addendum

Wellner; 2/28/2000

1. K 4.1, #9, page 239 and Bonus Chuck-a-luck. In the popular carnival game of chuck-a-luck, three fair dice are rolled. You bet on one of the numbers; say you bet a dollar on number 6. If no 6's show, you lose your dollar. If any 6's show, you get your dollar back, plus an extra dollar for each 6 that shows. Let $X$ be your net gain. So $X$ will be $-1$ if no 6's show; otherwise, it will be $+1$, $+2$, or $+3$ depending on the number of 6's that show.
(a) Find the probability mass function of $X$.
(b) Find $E(X)$.
(c) What do you expect to happen if you bet a dollar on 6 a thousand times?
**Additional questions not on the HW:**
(d) What is the variance of $X$?
(e) What is $P(T_n \geq 0$ where $T_n = X_1 + \cdots + X_n$ represents your winnings after $n$ plays?

**Solution:** (a) Let $T$ be the total number of 6's in the three rolls. Thus $T \sim$ Binomial$(n = 3, p = 1/6)$. Now $X = -1$ if $T = 0$, $X = t$ if $T = t$ for $t = 1, 2, 3$. Thus the probability mass function of $X$ is the same as that for $T$, namely $p_X(x) = \binom{3}{x}(1/6)^x(5/6)^{3-x}$ for $x = 1, 2, 3$, while $p_X(-1) = \binom{3}{0}(1/6)^0(5/6)^3$.
A simple way to say this is $X = T - 1_{[T=0]}$. (b) Based on the mass function for $X$ in (a) we compute

$$E(X) = \{(-1)(5/6)^3 + (1)(15/6^3) + (2)(15/6^3) + (3)(1/6^3)\} \doteq -.0787\,.$$

Another way to do the computation is via $X = T - 1_{[T=0]}$; then

$$E(X) = E(T) - E1_{[T=0]} = E(T) - P(T = 0) = 3 \cdot (1/6) - (5/6)^3 = -.0787\,.$$

(c) If we play the same game 1000 times, we expect to lose about \$78.70:

$$E(\sum_{i=1}^{1000} X_i) = \sum_{i=1}^{1000} E(X_i) \doteq (1000)(-.0787) \doteq -78.70\,.$$

As we calculated in class on Friday,

$$\begin{aligned} Var(X) &= Var(T - 1_{[T=0]}) = Var(T) - 2Cov(T, 1_{[T=0]}) + Var(1_{[T=0]}) \\ &= 3(1/6)(5/6) - 2\{E(T1_{[T=0]}) - E(T)P(T = 0)\} + (5/6)^3(1 - (5/6)^3) \\ &= 15/36 - 2\{0 - 3(1/6)(5/6)^3\} + (5/6)^3 - (5/6)^6 \\ &= 15/36 + 2(5/6)^3 - (5/6)^6 = 1.23918\,. \end{aligned}$$

Hence it follows that

$$Var(T_n) = Var(\sum_{i=1}^{1000} X_i) = nVar(X_1) = n(1.23918)\,.$$

By using the central limit theorem we can approximate

$$\begin{aligned} P(T_n \geq 0) &= P(T_n \geq -1/2) = P\left(\frac{T_n - n\mu}{\sqrt{n\sigma^2}} \geq \frac{-.5 - n\mu}{\sqrt{n\sigma^2}}\right) \\ &\doteq P\left(Z \geq \frac{-.5 - n\mu}{\sqrt{n\sigma^2}}\right). \end{aligned}$$

Here is a table of these means, variances, standard deviations, and probabilities of not losing after $n$ plays of the game of chuck-a-luck:

| $n$ | $E(T_n)$ | $\sigma^2_{T_n}$ | $\sigma_{T_n}$ | $P(T_n \geq 0)$ |
|---|---|---|---|---|
| 1 | -.0787 | 1.23918 | 1.1132 | 0.42 (E) |
| 10 | -.787 | 12.3918 | 3.5202 | ?? |
| 100 | - 7.87 | 123.918 | 11.132 | 0.254(A) |
| 1000 | -78.70 | 1239.18 | 35.202 | 0.013(A) |
| 10000 | -787.00 | 12391.8 | 111.318 | $1.37 \times 10^{-12}$(A) |

2. Bonus Problem 1: K 4.1, # 8, page 239. In part (c), let $T_n = X_1 + \cdots + X_n$ denote your winnings in $n$ plays of the game. Compute $E(T_n)$, $Var(T_n)$, and $\sigma_{T_n} = \sqrt{Var(T_n)}$ first for $n$ plays of the game; then make a table of these quantities for $n = 10^2, 10^3$, and $10^4$.

**Additional question not on the HW:**

What is $P(T_n \geq 0)$ where $T_n = X_1 + \cdots + X_n$ represents your winnings after $n$ plays?

**Solution:** For one play of the game, our winnings, $X_1$, has

$$E(X_1) = 1 \cdot \frac{18}{38} + (-1) \cdot \frac{20}{38} = -1/19 \doteq -0.05263158\,,$$

and variance

$$\begin{aligned} Var(X_1) &= \{1 - (-1/19)\}^2 \frac{18}{38} + \{-1 - (-1/19)\}^2 \frac{20}{38} \\ &= \left(\frac{20}{19}\right)^2 \cdot \frac{9}{19} + \left(\frac{18}{19}\right)^2 \cdot \frac{10}{19} \\ &= 6840/6859 \doteq 0.99723\,. \end{aligned}$$

Thus for $n$ plays of the game the expectation and variance of our total winnings $T_n = X_1 + \cdots + X_n$ are

$$E(T_n) = nE(X_1) = -n/19\,,$$

and

$$Var(T_n) = nVar(X_1) = n \cdot .99723\,.$$

Thus $\sigma_{T_n} = \sqrt{n} \cdot 0.998614$. By using the central limit theorem we can approximate

$$\begin{aligned} P(T_n \geq 0) &= P(T_n \geq -1/2) = P\left(\frac{T_n - n\mu}{\sqrt{n\sigma^2}} \geq \frac{-.5 - n\mu}{\sqrt{n\sigma^2}}\right) \\ &\doteq P\left(Z \geq \frac{-.5 - n\mu}{\sqrt{n\sigma^2}}\right). \end{aligned}$$

Here is a table of these means, variances, standard deviations, and probabilities of not losing after $n$ plays of the game of roulette:

| $n$ | $E(T_n)$ | $\sigma^2_{T_n}$ | $\sigma_{T_n}$ | $P(T_n \geq 0)$ |
|---|---|---|---|---|
| 1 | -0.0526 | .99723 | .9986 | 0.4737(E) |
| 10 | -.526 | 9.9723 | 3.158 | 0.5568(E) |
| 100 | - 5.26 | 99.723 | 9.99 | 0.3168(A) |
| 1000 | -52.63 | 997.23 | 31.58 | 0.0495(A) |
| 10000 | -526.32 | 9972.3 | 99.86 | $7.1135 \times 10^{-8}$(A) |

Note that since $T_n = 2W_n - n$ where $W_n \sim \text{Binomial}(n, 18/38)$, $P(T_{10} = 0) = P(W_{10} = 5) = 0.2427$, so $P(T_{10} > 0) = .3141$.