Statistics 394, Problem Set 7 Solutions

Wellner; 2/23/2000

1. K 3.5, #9, page 207. Suppose the joint density function of X and Y is $f(x,y) = 27y^2x^{-2}e^{-3x}1_{[0<y<x<\infty]}$.
(a) Find the marginal density for X.
(b) Are X and Y independent? Why or why not?

Solution: (a) The marginal density of X is

$$\begin{aligned} f_X(x) &= \int f(x,y)dy = \int_0^x 27y^2x^{-2}e^{-3x}\,dy \\ &= 27x^{-2}e^{-3x}\int_0^x y^2\,dy \\ &= 27x^{-2}e^{-3x}(1/3)x^3 \\ &= 9xe^{-3x}\,, \qquad \text{for} \qquad 0 < x < \infty\,. \end{aligned}$$

Note that this is the Gamma$(2,3)$ density; i.e. the distribution of the waiting time until the second event in a Poisson process with intensity $\nu = 3$.
(b) No. The marginal density f_Y of Y is something positive for $0 < y < \infty$, and this yields, for $0 < y < x < \infty$,

$$f_X(x)f_Y(y) \neq 0 = f(x,y)\,,$$

contradicts independence of X and Y. [In fact, the marginal density of Y is

$$\begin{aligned} f_Y(y) &= \int f(x,y)dx = \int_y^\infty 27y^2x^{-2}e^{-3x}\,dx \\ &= 27y^2\int_y^\infty x^{-2}e^{-3x}dx \\ &= 27y^2\{y^{-1}e^{-3y} - 3\int_y^\infty t^{-1}e^{-t}dt\}\,. \end{aligned}$$

The integral appearing in this last expression cannot be simplified much further; this is the *incomplete Gamma function* at $(0,y)$.]

2. K 4.1, #3, page 238. Find $E(X)$ if X has the given density:
(a) $f(x) = 6x(1-x)1_{(0,1)}(x)$.
(b) $f(x) = (3/x^4)1_{(1,\infty)}(x)$.

Solution. (a) $E(X) = \int_0^1 x6x(1-x)dx = 6(\int_0^1 x^2dx - \int_0^1 x^3dx) = 6(1/3-1/4) = 6/12 = 1/2$.
(b) $E(X) = \int_1^\infty x(3/x^4) = \int_1^\infty (3/x^3)dx = 3/2$.

3. K 4.1, #5, page 238. A hat contains five slips of paper numbered $1, 2, 3, 4, 5$. Two are chosen at random, without replacement, and X is the sum of the two numbers. Write down the possible vlaues of X and their probabilities; from them find EX.

   Solution: Denote the number drawn on the ith draw by Y_i, $i = 1, 2$. The the possible values for (Y_1, Y_2), and hence of $X = Y_1 + Y_2$ are as shown in the following table:

Y_1	Y_2	X
1	2	3
1	3	4
1	4	5
1	5	6
2	1	3
2	3	5
2	4	6
2	5	7
3	1	4
3	2	5
3	4	7
3	5	8
4	1	5
4	2	6
4	3	7
4	5	9
5	1	6
5	2	7
5	3	8
5	4	9

   Each of these outcomes has probability $1/20$. Thus the probability distribution of X is given by the following table:

x	3	4	5	6	7	8	9	
$20 \cdot p(x)$	2	2	4	4	4	2	2	20
$20 \cdot x \cdot p(x)$	6	8	20	24	28	16	18	120

   It follows that $E(X) = 120/20 = 6$. [Note that this also follows from $E(X) = E(Y_1) + E(Y_2) = \overline{a} + \overline{a} = 3 + 3 = 6$ since $\overline{a} = (1 + 2 + 3 + 4 + 5)/5 = 5 \cdot 6/10 = 3$.]

4. K 4.1, #7, page 238. The mass function of X is given by $p(k) = (k-1)(1/2)^k$ for $k = 2, 3, \ldots$. Find $E(X)$.

   Solution.

$$\begin{aligned} E(X) &= \sum_{k=2}^{\infty} k(k-1)(1/2)^k = (1/2)^2 \frac{d^2}{dp^2} \sum_{k=0}^{\infty} p^k \Big|_{p=1/2} \\ &= (1/2)^2 \frac{d^2}{dp^2} \frac{1}{1-p} \Big|_{p=1/2} \end{aligned}$$

$$
\begin{aligned}
&= (1/2)^2 \frac{d}{dp}\frac{-1}{(1-p)^2}\Big|_{p=1/2} \\
&= (1/2)^2 \frac{2}{(1-p)^3}\Big|_{p=1/2} \\
&= (1/2)\frac{1}{(1/2)^3}\Big|_{p=1/2} \\
&= 4\,.
\end{aligned}
$$

[Note that the waiting time W_2 until the second success in Bernoulli trials with probability $p = 1/2$ of success on each trial has a Negative Binomial$(2, 1/2)$ distribution with probability mass function

$$P(W_2 = k) = \binom{k-1}{1}(1/2)^{k-2}(1/2)^2 = \binom{k-1}{1}(1/2)^k, \qquad k = 2, 3, \ldots .$$

Thus X has the same distribution as $W_2 \sim$ Negative Binomial$(2, 1/2)$. We know that the expectation of $W_2 \sim$Negative Binomial$(2, p)$ is $2/p$, so with $p = 1/2$, this becomes $2/(1/2) = 4$.]

5. K 4.1, #9, page 239. Chuck-a-luck. In the popular carnival game of chuck-a-luck, three fair dice are rolled. You bet on one of the numbers; say you bet a dollar on number 6. If no 6's show, you lose your dollar. If any 6's show, you get your dollar back, plus an extra dollar for each 6 that shows. Let X be your net gain. So X will be -1 if no 6's show; otherwise, it will be $+1$, $+2$, or $+3$ depending on the number of 6's that show.
(a) Find the probability mass function of X.
(b) Find $E(X)$.
(c) What do you expect to happen if you bet a dollar on 6 a thousand times?

Solution: (a) Let T be the total number of 6's in the three rolls. Thus $T \sim$ Binomial$(n = 3, p = 1/6)$. Now $X = -1$ if $T = 0$, $X = t$ if $T = t$ for $t = 1, 2, 3$. Thus the probability mass function of X is the same as that for T, namely $p_X(x) = \binom{3}{x}(1/6)^x(5/6)^{3-x}$ for $x = 1, 2, 3$, while $p_X(-1) = \binom{3}{0}(1/6)^0(5/6)^3$.
A simple way to say this is $X = T - 1_{[T=0]}$. (b) Based on the mass function for X in (a) we compute

$$E(X) = \{(-1)(5/6)^3 + (1)(15/6^3) + (2)(15/6^3) + (3)(1/6^3)\} \doteq -.0787\,.$$

Another way to do the computation is via $X = T - 1_{[T=0]}$; then

$$E(X) = E(T) - E1_{[T=0]} = E(T) - P(T = 0) = 3 \cdot (1/6) - (5/6)^3 = -.0787\,.$$

(c) If we play the same game 1000 times, we expect to lose about \$78.70:

$$E(\sum_{i=1}^{1000} X_i) = \sum_{i=1}^{1000} E(X_i) \doteq (1000)(-.0787) \doteq -78.70\,.$$

6. K 4.1, #15, page 240. If X has the density $f(x) = xe^{-x^2/2}1_{(0,\infty)}(x)$, find $E(X)$.

Solution: For this density,

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_0^{\infty} xxe^{-x^2/2}dx \\ &= \int_0^{\infty} udv \qquad \text{with} \;\; u = x, \;\; dv = xe^{-x^2/2}dx\,, \\ &= uv|_0^{\infty} - \int_0^{\infty} vdu \;\; \text{since} \;\; du = dx, \; v = -e^{-x^2/2} \\ &= -xe^{-x^2/2}|_0^{\infty} + \int_0^{\infty} e^{-x^2/2}dx \\ &= 0 + \sqrt{2\pi}\int_0^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\ &= \sqrt{2\pi}\int_0^{\infty} \phi(x)dx \\ &= \sqrt{2\pi}\frac{1}{2} \end{aligned}$$

since for a standard normal random variable Z with density $\phi(z) = e^{-z^2/2}/\sqrt{2\pi}$ we have

$$P(Z \geq 0) = \int_0^{\infty} \phi(z)dz = 1/2\,.$$

Alternatively, for this density,

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_0^{\infty} xxe^{-x^2/2}dx \\ &= \sqrt{2\pi}\int_0^{\infty} x^2\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\ &= \sqrt{2\pi}\int_0^{\infty} x^2\phi(x)dx \\ &= \frac{\sqrt{2\pi}}{2}\int_{-\infty}^{\infty} x^2\phi(x)dx \\ &= \frac{\sqrt{2\pi}}{2} \end{aligned}$$

since $\phi(z) = e^{-z^2/2}/\sqrt{2\pi}$ is the standard normal, $N(0,1)$, density and since $\int_{-\infty}^{\infty} x^2\phi(x)dx = Var(Z) = 1$ for a standard normal random variable Z with density ϕ. [Another way to do this is as follows: by the change of variables $y = x^2/2$,

$$\begin{aligned} E(X) &= \int_0^{\infty} xxe^{-x^2/2}dx \\ &= \int_0^{\infty} \sqrt{2}y^{1/2}e^{-y}\,dy \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{2}\Gamma(3/2) = \sqrt{2}\frac{1}{2}\Gamma(1/2) \\
&= \sqrt{2\pi}/2
\end{aligned}
$$

since the Gamma function $\Gamma(r) \equiv \int_0^\infty y^{r-1}e^{-y}dy$ satisfies $\Gamma(r+1) = r\Gamma(r)$ and $\Gamma(1/2) = \sqrt{\pi}$; see pages 490 and 491.]

7. Bonus Problem 1: K 4.1, # 8, page 239. In part (c), let $T_n = X_1 + \cdots + X_n$ denote your winnings in n plays of the game. Compute $E(T_n)$, $Var(T_n)$, and $\sigma_{T_n} = \sqrt{Var(T_n)}$ first for n plays of the game; then make a table of these quantities for $n = 10^2, 10^3$, and 10^4.

   Solution: For one play of the game, our winnings, X_1, has

   $$E(X_1) = 1 \cdot \frac{18}{38} + (-1) \cdot \frac{20}{38} = -1/19 \doteq -0.05263158\,,$$

   and variance

   $$
   \begin{aligned}
   Var(X_1) &= \{1-(-1/19)\}^2\frac{18}{38} + \{-1-(-1/19)\}^2\frac{20}{38} \\
   &= \left(\frac{20}{19}\right)^2 \cdot \frac{9}{19} + \left(\frac{18}{19}\right)^2 \cdot \frac{10}{19} \\
   &= 6840/6859 \doteq 0.99723\,.
   \end{aligned}
   $$

   Thus for n plays of the game the expectation and variance of our total winnings $T_n = X_1 + \cdots + X_n$ are

   $$E(T_n) = nE(X_1) = -n/19\,,$$

   and

   $$Var(T_n) = nVar(X_1) = n \cdot .99723\,.$$

   Thus $\sigma_{T_n} = \sqrt{n} \cdot 0.998614$. Here is a table of these means and standard deviations:

n	$E(T_n)$	σ_{T_n}
100	- 5.26	9.99
1000	-52.63	31.58
10000	-526.32	99.86

8. Bonus Problem 2: Look at the Bivariate Uniform Experiment at the Virtual Laboratories website:

   `http://www.math.uah.edu/stat/expect/index.html`

   (a) For the circle part of this experiment, give a picture showing the region where the density is positive and where it is zero.
   (b) For the circle part of this experiment, find the marginal density of X.
   (c) Find the conditional density of $(Y|X = x)$ for x in the range -6 to 6.
   (d) Verify that the correlation coefficient equals zero: $\rho_{X,Y} = 0$.

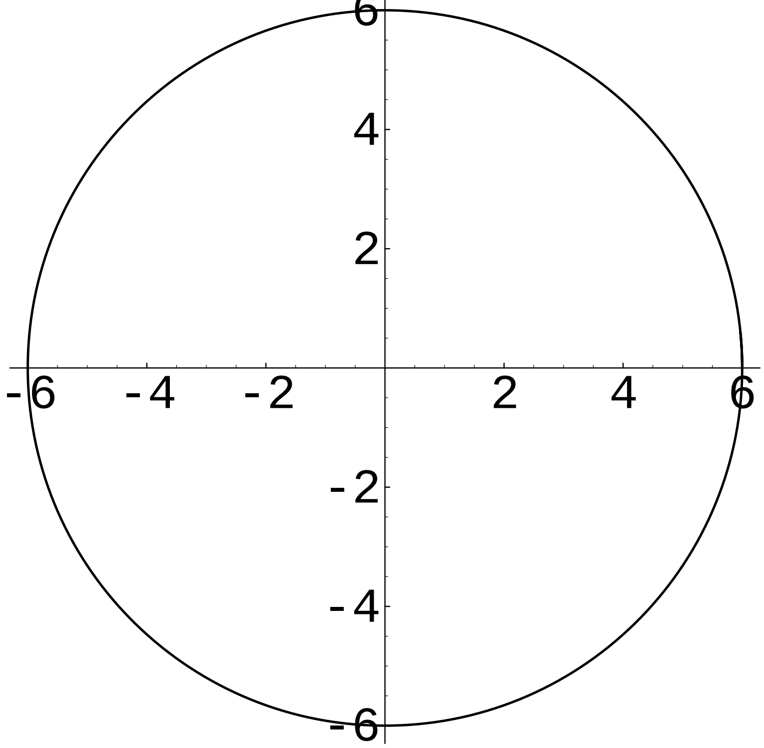

Figure 1: Positivity Region for Problem 8.

Solution: (a) The region where the density is positive is the interior of the circle centered at the origin with radius 6

$$C = \{(x,y) \in R^2 : x^2 + y^2 \le 6^2\};$$

see Figure 1.

(b) The marginal density f_X of X is given by

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_{-\sqrt{6^2-x^2}}^{\sqrt{6^2-x^2}} \frac{1}{36\pi^2} dy \\ &= \frac{1}{36\pi^2} 2\sqrt{6^2 - x^2} 1_{(-6,6)}(x)\,. \end{aligned}$$

Note that

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-6}^{6} \frac{1}{36\pi^2} 2\sqrt{6^2 - x^2}\, dx = \frac{36\pi^2}{36\pi^2} = 1\,.$$

(c) The conditional density $f_{Y|X}(y|x)$ of $(Y|X = x)$ is

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} \\ &= \frac{(1/36\pi^2)}{\sqrt{6^2 - x^2}/18\pi^2} 1_{(-\sqrt{6^2-x^2}, +\sqrt{6^2-x^2})}(y) \\ &= \frac{1}{2\sqrt{6^2 - x^2}} \end{aligned}$$

for $-6 < x < 6$, and 0 otherwise; i.e. $(Y|X = x) \sim$Uniform$(-\sqrt{6^2 - x^2}, \sqrt{6^2 - x^2})$.
(d) It is clear that the marginal density of Y is the same as that for X, and that $E(X) = 0 = E(Y)$. Thus $Cov(X, Y) = E(XY)$, and the latter is

$$\begin{aligned} E(XY) &= \int\int_C xyf(x, y)dxdy \\ &= \int\int_C xy\frac{1}{36\pi^2}dxdy = 0 \end{aligned}$$

by symmetry. Hence $Cov(X, Y) = 0$, and also $\rho_{X,Y} = 0$. Note that this agrees with the assertion in the circular part of this experiment.