

Statistics 394, Problem Set 5 Solutions

Wellner; 2/9/2000

1. K 3.1, # 7 (page 155). The possible values of a certain discrete distribution are 1, 2, 3, 4, and 5, and the probability assigned to a possible value x is proportional to x^2 .

(a) What is the mass function of this distribution?

(b) What probability does this distribution assign to the interval $(1, 3]$?

Solution: (a) $P(X = x) = cx^2$ for $x \in \{1, \dots, 5\}$, so

$$1 = \sum_{x=1}^5 P(X = x) = \sum_{x=1}^5 cx^2 = c(1 + 4 + 9 + 16 + 25) = 55c,$$

so $c = 1/55$.

(b) $P(1 < X \leq 3) = P(X = 2) + P(X = 3) = (4 + 9)/55 = 13/55$.

For K3.1, #8 (page 155), the reasoning goes as follows: (a) The density $f_X(x) = cx^2 1_{[1,5]}(x)$, so

$$1 = \int_1^5 f_X(x) dx = \int_1^5 cx^2 dx = \frac{c}{3}(x^3|_1^5) = \frac{c}{3}(5^3 - 1^3) = \frac{124}{3}c,$$

so $c = 3/124$, and $f_X(x) = (3/124)x^2 1_{[1,5]}(x)$. Then

$$P(1 < X \leq 3) = \int_1^3 (3/124)x^2 dx = (1/124)(3^3 - 1) = 26/124 = 13/62.$$

2. K 3.2, # 1 (page 167). Write down the CDF of the Binomial distribution with $n = 2$ and $p = 1/6$ in the same form that was used at the end of Example 3.2.2. Also sketch the graph.

Solution: For $X \sim \text{Binomial}(2, 1/6)$, $P(X = 0) = (5/6)^2 = 25/36$, $P(X = 1) = 2(1/6)(5/6) = 10/36$, and $P(X = 2) = (1/6)^2 = 1/36$. Thus the CDF is

$$F(x) = P(X \leq x) = \begin{cases} 0, & x < 0 \\ 25/36, & 0 \leq x < 1 \\ 35/36, & 1 \leq x < 2 \\ 36/36 = 1, & 2 \leq x < \infty. \end{cases}$$

See Figure 1 for a plot of F .

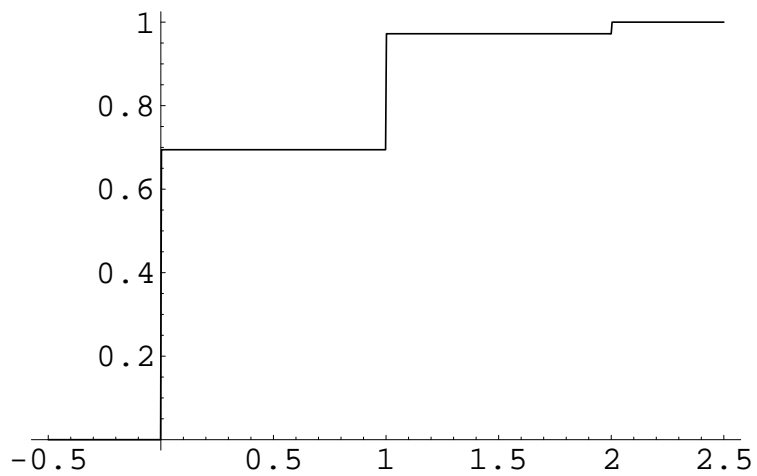


Figure 1: Plot of Binomial(2, 1/6) CDF.

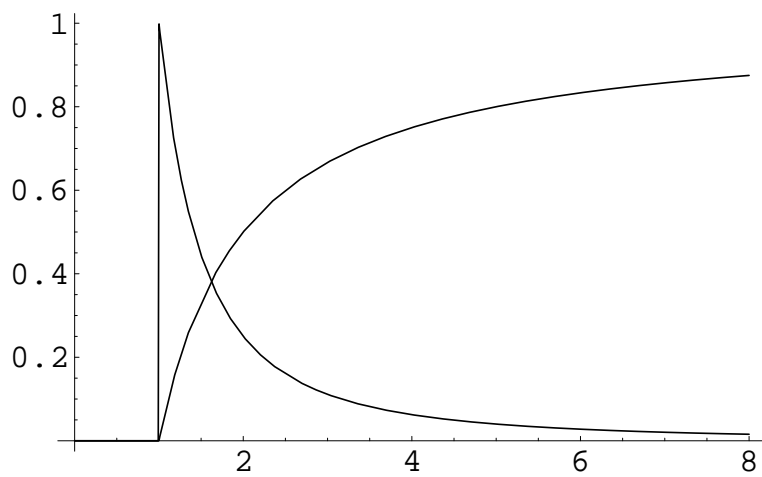


Figure 2: Plot of density f and CDF F .

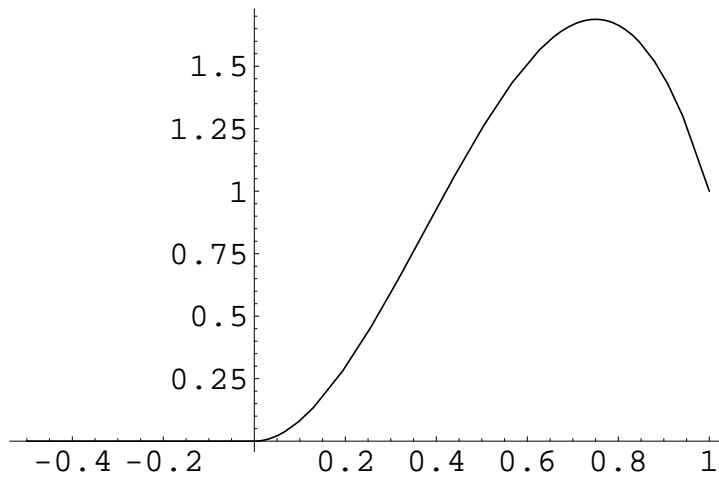


Figure 3: Plot of density f .

3. K 3.2, # 4 (page 167). Give the CDF of the distribution whose density is defined by $f(x) = x^{-2}1_{[1,\infty)}(x)$.

Solution: For $x < 1$, $F(x) = P(X \leq x) = 0$, while for $x \geq 1$,

$$F(x) = P(X \leq x) = \int_{-\infty}^x f_X(y)dy = \int_1^x y^{-2}dy = -y^{-1}|_1^x = 1 - 1/x.$$

See Figure 2 for a plot of both f and F .

4. K 3.2, # 5 (page 167). Find a density for the distribution whose CDF is

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ 3x^3 - 2x^4, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } 1 \leq x < \infty. \end{cases}$$

Solution: The corresponding density is $f = F'$: here we compute

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ 9x^2 - 8x^3, & \text{if } 0 \leq x < 1, \\ 0, & \text{if } 1 \leq x < \infty \end{cases} = (9x^2 - 8x^3)1_{[0,1)}(x).$$

See figure 3 for a plot of f and Figure 4 for a plot of the cdf F .

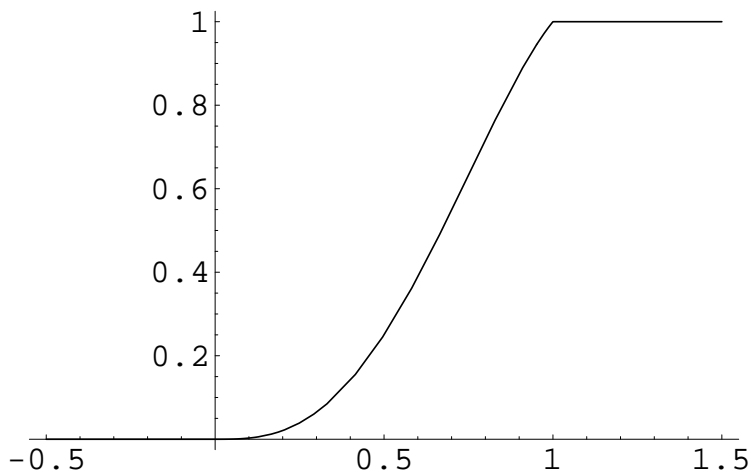


Figure 4: Plot of CDF F .

5. Suppose that two fair dice are rolled and the sequence of scores (X_1, X_2) are recorded. Let $T = X_1 + X_2$ and $D = X_1 - X_2$ denote the sum and difference of the scores, respectively.
- Find the joint mass function of (T, D) ; call it $p_{T,D}$.
 - Find the marginal probability mass function of D ; call it p_D .
 - Find the marginal probability mass function of T ; call it p_T .
 - Are Y and Z independent? (Justify your answer.)

Solution: If we record the values of (T, D) for each possible outcome, we get the following table:

x_2/x_1	1	2	3	4	5	6
6	(7,-5)	(8,-4)	(9,-3)	(10,-2)	(11,-1)	(12,0)
5	(6,-4)	(7,-3)	(8,-2)	(9,-1)	(10,0)	(11,1)
4	(5,-3)	(6,-2)	(7,-1)	(8,0)	(9,1)	(10,2)
3	(4,-2)	(5,-1)	(6,0)	(7,1)	(8,2)	(9,3)
2	(3,-1)	(4,0)	(5,1)	(6,2)	(7,3)	(8,4)
1	(2,0)	(3,1)	(4,2)	(5,3)	(6,4)	(7,5)

This yields the following joint and marginal distributions for (T, D) : the entries in the table are $36 \cdot p_{T,D}(t, d)$, $36 \cdot p_D(d)$, and $36 \cdot p_T(t)$.

d/t	2	3	4	5	6	7	8	9	10	11	12	$p_D(d)$
5						1						1
4					1		1					2
3				1		1		1				3
2			1		1		1		1			4
1		1		1		1		1		1		5
0	1		1		1		1		1		1	6
-1		1		1		1		1		1		5
-2			1		1		1		1			4
-3				1		1		1				3
-4					1		1					2
-5						1						1
$p_T(t)$	1	2	3	4	5	6	5	4	3	2	1	36

6. Suppose the same setup as in the previous problem.
- (a) Find the conditional probability mass function of T given $D = d \in \{-5, -4, \dots, 5\}$.
- (b) Find the conditional probability mass function of D given $T = t \in \{2, 3, \dots, 12\}$.

Solution: For the joint distribution of (T, M) in Example 2, We easily compute

$$P(D = d|T = 7) = \begin{cases} \frac{1/36}{6/36} = 1/6, & d \in \{-5, -3, -1, 1, 3, 5\} \\ 0, & \text{otherwise} \end{cases},$$

$$P(D = d|T = 6) = P(D = d|T = 8) = \begin{cases} \frac{1/36}{5/36} = 1/5, & d \in \{-4, -2, 0, 2, 4\} \\ 0, & \text{otherwise} \end{cases},$$

$$P(D = d|T = 5) = P(D = d|T = 9) = \begin{cases} \frac{1/36}{4/36} = 1/4, & d \in \{-3, -1, 1, 3\} \\ 0, & \text{otherwise} \end{cases},$$

$$P(D = d|T = 4) = P(D = d|T = 10) = \begin{cases} \frac{1/36}{3/36} = 1/3, & d \in \{-2, 0, 2\} \\ 0, & \text{otherwise} \end{cases},$$

$$P(D = d|T = 3) = P(D = d|T = 11) = \begin{cases} \frac{1/36}{2/36} = 1/2, & d \in \{-1, 1\} \\ 0, & \text{otherwise} \end{cases},$$

$$P(D = d|T = 2) = P(D = d|T = 12) = \begin{cases} \frac{1/36}{1/36} = 1, & d = 0 \\ 0, & \text{otherwise} \end{cases},$$

Conditioning on D rather than T , we find:

$$P(T = t|D = 0) = \begin{cases} \frac{1/36}{6/36} = 1/6, & t \in \{2, 4, \dots, 12\} \\ 0, & \text{otherwise} \end{cases},$$

$$P(T = t|D = \pm 1) = \begin{cases} \frac{1/36}{5/36} = 1/5, & t \in \{3, 5, \dots, 11\} \\ 0, & \text{otherwise} \end{cases},$$

$$P(T = t|D = \pm 2) = \begin{cases} \frac{1/36}{4/36} = 1/4, & t \in \{4, 6, \dots, 10\} \\ 0, & \text{otherwise} \end{cases},$$

$$P(T = t|D = \pm 3) = \begin{cases} \frac{1/36}{3/36} = 1/3, & t \in \{5, 7, 9\} \\ 0, & \text{otherwise} \end{cases},$$

$$P(T = t|D = \pm 4) = \begin{cases} \frac{1/36}{2/36} = 1/2, & t \in \{6, 8\} \\ 0, & \text{otherwise} \end{cases},$$

$$P(T = t|D = \pm 5) = \begin{cases} \frac{1/36}{1/36} = 1, & t = 7 \\ 0, & \text{otherwise} \end{cases},$$