Statistics 394, Problem Set 4 Solutions

Wellner; 2/2/2000

1. K 1.4, # 6; compare with the answers to 1.4, #5 from Problem Set 3.

   Solution: Let's agree to measure time in years; then the rate for the Poisson process is $\nu = 1/4$ per year.
   (a) Since $\mathbb{N}(4) \sim Poisson(\lambda = \nu t = (1/4)(4) = 1)$,

   $$P(\mathbb{N}(4) = 0) = e^{-1} \doteq .3679\,.$$

   (b) Since $Y_1 \sim Exponential(\nu = 1/4)$, we have

   $$P(Y_1 \le 1) = 1 - P(Y_1 > 1) = 1 - e^{-1/4} = .2212\,.$$

   (c) Since $\mathbb{N}(1) \sim Poisson(\lambda = \nu t = (1/4)(1) = 1/4)$, and $\mathbb{N}(2) - \mathbb{N}(1) \sim Poisson(1/4)$ and is independent of $\mathbb{N}(1)$,

   $$\begin{aligned} P(\mathbb{N}(1) = 0, \mathbb{N}(2) - \mathbb{N}(1) \ge 1) &= P(\mathbb{N}(1) = 0)P(\mathbb{N}(2) - \mathbb{N}(1) \ge 1) \\ &= e^{-1/4}(1 - P(\mathbb{N}(2) - \mathbb{N}(1) = 0)) \\ &= e^{-1/4}(1 - e^{-1/4}) = .1723\,. \end{aligned}$$

2. K 1.4, # 7:

   Solution: Here $\nu = .9$ tornadoes per week.
   (a) Now $Y_1 \sim Exponential(\nu = .9)$, so $P(Y_1 > 2) = \exp(-\nu 2) = \exp(-1.8) \doteq 0.1653\,.$
   (b) Since Y_2 also has an exponential distribution with parameter $\nu = .9$, $P(Y_2 \le 2/7) = 1 - P(Y_2 > 2/7) = 1 - \exp(-\nu(2/7)) = 0.2267.$
   (c) Since the distribution of the number of tornadoes is the same over any two week period, and $\mathbb{N}(2) \sim Poisson(.9 \times 2 = 1.8)$, the probability in question is the same as

   $$\begin{aligned} P(\mathbb{N}(2) \ge 3) &= 1 - P(\mathbb{N}(2) \le 2) \\ &= 1 - e^{-1.8} - e^{-1.8}\frac{1.8}{1!} - e^{-1.8}\frac{(1.8)^2}{2!} \\ &\doteq 1 - 0.7306 = .2694\,. \end{aligned}$$

3. K 1.4, # 13.

   Solution: (I did this one in class on 1/28, but here it is again.) Here $\nu = 4$ and $\mathbb{N}(1) \sim Poisson(\lambda = \nu t = (4)(1) = 4)$. The fact that we summarize events in the

Poisson process according to 1 minute time intervals means that we are back in the setting of a Bernoulli process with p, the probability of "success" being

$$p = P(\mathbb{N}(1) \geq 2) = 1 - P(\mathbb{N}(1) \leq 1) = 1 - e^{-4} - e^{-4}\frac{4}{1!} = 1 - 5e^{-4} \doteq .9084 .$$

(a) Now the total number of successful one-minutes time intervals in 10 trials is T_{10} ~Binomial$(10, p = .9084)$. Hence

$$P(T_{10} = 8) = \binom{10}{8} p^8 (1-p)^2 = .1750 .$$

(b) $P(T_{10} = 10) = \binom{10}{10} p^{10}(1-p)^0 = .3827$.
(c)

$$\begin{aligned} P(T_{10} \geq 8) &= P(T_{10} = 8) + P(T_{10} = 9) + P(T_{10} = 10) \\ &= .3827 + .3858 + .1750 = .9435 . \end{aligned}$$

4. Customers arrive at a bank at a Poisson rate λ. Suppose two customers arrived during the first hour. What is the probability that:
(a) Both arrived during the first 20 minutes?
(b) at least one arrived during the first 20 minutes?

Solution: Here we want to condition on $\mathbb{N}(1) = 2$.
Note that $\mathbb{N}(1) - \mathbb{N}(1/3) \sim Poisson(2\lambda/3)$ and is independent of $\mathbb{N}(1/3)$. (a) Now

$$\begin{aligned} P(\mathbb{N}(1/3) = 2 | \mathbb{N}(1) = 2) &= \frac{P(\mathbb{N}(1/3) = 2, \mathbb{N}(1) = 2)}{P(\mathbb{N}(1) = 2)} \\ &= \frac{P(\mathbb{N}(1/3) = 2, \mathbb{N}(2) - \mathbb{N}(1/3) = 0)}{P(\mathbb{N}(1) = 2)} \\ &= \frac{P(\mathbb{N}(1/3) = 2) P(\mathbb{N}(2) - \mathbb{N}(1/3) = 0)}{P(\mathbb{N}(1) = 2)} \\ &= \frac{e^{-\lambda/3}\frac{(\lambda/3)^2}{2!} e^{-2\lambda/3}\frac{(2\lambda/3)^0}{0!}}{e^{-\lambda}\frac{\lambda^2}{2!}} \\ &= (1/3)^2 = 1/9 . \end{aligned}$$

(b) We want to compute

$$\begin{aligned} P(\mathbb{N}(1/3) \geq 1 | \mathbb{N}(1) = 2) &= 1 - P(\mathbb{N}(1/3) = 0 | \mathbb{N}(1) = 2) \\ &= 1 - \frac{P(\mathbb{N}(1/3) = 0, \mathbb{N}(1) = 2)}{P(\mathbb{N}(1) = 2)} \\ &= 1 - \frac{P(\mathbb{N}(1/3) = 0, \mathbb{N}(1) - \mathbb{N}(1/3) = 2)}{P(\mathbb{N}(1) = 2)} \\ &= 1 - \frac{P(\mathbb{N}(1/3) = 0) P(\mathbb{N}(1) - \mathbb{N}(1/3) = 2)}{P(\mathbb{N}(1) = 2)} \\ &= 1 - \frac{e^{-\lambda/3} \cdot e^{-2\lambda/3}\frac{(2\lambda/3)^2}{2!}}{e^{-\lambda}\frac{\lambda^2}{2!}} \\ &= 1 - (2/3)^2 = 1 - (4/9) = 5/9 . \end{aligned}$$

5. Cars cross a certain point in the highway in accordance with a Poisson process with rate $\nu = 3$ per minute. If Al blindly runs across the highway, then what is the probability that he will be uninjured if the amount of time that it takes him to cross the road is s seconds? (Assume that if he is on the highway when a car passes by, then he will be injured.) Do it for $s = 2, 5, 10, 20$.

Solution: Here $\lambda = \nu s$ with $\nu = 3$ per minute. We want to compute $P(\mathbb{N}(s) = 0) = e^{-3s}$ for $s = 2/60, 5/60, 10/60, 20/60$. Here's a table of the resulting probabilities:

s	$P(\mathbb{N}(s) = 0)$
2	$e^{-6/60} = 0.9048$
5	$e^{-15/60} = 0.7788$
10	$e^{-30/60} = 0.6065$
20	$e^{-60/60} = 0.3679$

6. Suppose in the preceding problem that Al is agile enough to escape from a single car, but if he encounters two or more cars while attempting to cross the road, then he is injured. What is the probability that he will be unhurt if it takes him s seconds to cross? Do it for $s = 2, 5, 10, 20, 30$.

Solution: As in the previous problem, $\lambda = \nu s$ with $\nu = 3$ per minute. We want to compute $P(\mathbb{N}(s) \leq 1) = e^{-3s}(1 + 3s)$ for $s = 2/60, 5/60, 10/60, 20/60, 30/60$. Here's a table of the resulting probabilities:

s	$P(\mathbb{N}(s) \leq 1)$
2	$e^{-6/60}(1 + 6/60) = 0.9953$
5	$e^{-15/60}(1 + 15/60) = 0.9735$
10	$e^{-30/60}(1 + 30/60) = 0.9098$
20	$e^{-60/60}(1 + 60/60) = 0.7358$
30	$e^{-90/60}(1 + 90/60) = 0.5578$

7. Bonus Problem: Look at the Poisson Experiment at the Virtual Laboratories website:

`http://www.math.uah.edu/stat/poisson/index.html` .

(a) For $\nu = r = 1$, and $t = 2.5$, Verify the probabilities $P(\mathbb{N}(2.5) = k)$ you see plotted in blue: for $k = 0, 1, 2$.
(b) What is the relationship between these probabilities (for $k = 1, 2, \ldots$) and the probabilities $P(W_m > t)$ for $m = 1, 2, \ldots$ and appropriate t?

Solution: (a) $P(\mathbb{N}(2.5) = k) = e^{-2.5}(2.5)^k/k!$ for $k = 0, 1, 2$. I calculate these to be

$$\begin{aligned} P(\mathbb{N}(2.5) = 0) &= e^{-2.5} = 0.0821 \\ P(\mathbb{N}(2.5) = 1) &= (2.5)e^{-2.5} = 0.2052 \\ P(\mathbb{N}(2.5) = 2) &= ((2.5)^2/2)e^{-2.5} = 0.2565\,. \end{aligned}$$

This is in agreement with the plot of the Poisson mass function shown at the web site with $\nu = r = 1$ and $t = 2.5$.
(b) The basic identity for a Poisson process says that $[W_m > t] = [\mathbb{N}(t) < m]$. Hence (assuming that $\nu = 1$)

$$P(W_m > t) = P(\mathbb{N}(t) < m) = \sum_{k=0}^{m-1} P(\mathbb{N}(t) = k) = \sum_{k=0}^{m-1} e^{-t} \frac{t^k}{k!}$$

for any t and m.