Statistics 394, Problem Set 3 Solutions

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1. K 2.3, # 1: In a certain population of voters, 80% of the Democrats, 45% of the Republicans, and 55% of the independents say the president is doing a good job. The voters are 35% Democrat, 40% Republican, and 25% independent.
(a). What proportion of the voters thnk the president is doing a good job?
(b). A randomly chosen voter thinks the president is doing a good job. What is the probability that the voter is a Democrat?

   Solution: (a) Let G be the event that a randomly chosen voter thinks the president is doing a good job, and let D, R, I be the events that such a voter is a Democrat, Republican, or Independent, respectively. Then we have $P(G|D) = .8$, $P(G|R) = .45$, and $P(G|I) = .55$. Also, $P(D) = .35$, $P(R) = .40$, and $P(I) = .25$. Hence

$$\begin{aligned} P(G) &= P(G\cap D) + P(G\cap R) + P(G\cap I) \\ &= P(G|D)P(D) + P(G|R)P(R) + P(G|I)P(I) \\ &= (.8)(.35) + (.45)(.4) + (.55)(.25) \\ &= .5975\,. \end{aligned}$$

   (b) This is easily calculated as

$$P(D|G) = \frac{P(D\cap G)}{P(G)} = \frac{P(G|D)P(D)}{P(G)} = \frac{(.8)(.35)}{.5975} = .4686\,.$$

2. K 2.2, # 3: Two of the integers form 1 to 100 are chosen at random, one at a time, without replacement. Find the following:
(a) The probability that 1 is chosen.
(b) The conditional probability that 1 is chosen, given that 2 is not chosen.
(c) The conditional probability that 1 is the second number chosen, given neither 1 nor 2 is the first.
(d) The conditional probability that 1 is the second number chosen given that 2 is not the first.

   Solution: (a) Suppose that the number 1 is considered as the one "red ball" in an urn containing 100 balls; thus the numbers $2, \ldots, 100$ are considered to be the white balls. Then with $T_2 \equiv$the number of red balls drawn in two draws,

$$P(T_2 = 1) = \frac{\binom{1}{1}\binom{99}{1}}{\binom{100}{2}} = \frac{1\cdot 99}{100\cdot 99/2} = 2/100 = .02\,.$$

   Note that we can also compute this as $P(1 \text{ drawn on first draw}) + P(1 \text{ drawn on second draw}) =$

$$\frac{1}{100}\cdot\frac{99}{99} + \frac{99}{100}\cdot\frac{1}{99} = \frac{2}{100}\,.$$

(b) The probability that 1 is chosen, given that 2 is not chosen, Since we are conditioning on 2 not being chosen, we can revisalize the experiment as consisting of an urn with just 99 balls: the number 1 and the numbers $3, \cdots, 100$. Then as in the second solution of part (a) the conditional probability of drawing the 1 is

$$\frac{1}{99} \cdot \frac{98}{98} + \frac{98}{99} \cdot \frac{1}{98} = \frac{2}{99} .$$

Alternatively, thinking of the urn as containing three types of balls (the 1, the 2, and all the others), the probability of getting the 1 and not the 2 is

$$\frac{\binom{1}{1} \cdot \binom{1}{0} \cdot \binom{98}{1}}{\binom{100}{2}} = \frac{98}{100 \cdot 99/2} = \frac{2 \cdot 98}{100 \cdot 99} .$$

Since the probability of not drawing the 2 is $1 - 2/100 = 98/100$, it follows that the conditional probability of drawing the 1 given that the 2 is not drawn is

$$\frac{(2 \cdot 98)/(100 \cdot 99)}{98/100} = \frac{2}{99} .$$

(c) The conditional probability of drawing the 1 on the second draw, given that the first number drawn is neither the 1 nor 2 is just

$$\frac{(98/100) \cdot (1/99)}{(98/100)} = 1/99 .$$

(d) The conditional probability that 1 is the second number chosen given that 2 is not the first is just

$$\frac{(98/100) \cdot (1/99)}{99/100} = \frac{98}{99^2} = \frac{98}{9801} \approx .00999898 .$$

3. K 2.2, # 10: In the standard normal probability model, find the following conditional probabilities:
(a) $P(Z < 1|Z < 2)$.
(b) $P(Z < 1|Z > 0)$.
(c) $P(Z > 0|Z < 1)$.

Solutions: (a) $P(Z < 1|Z < 2) = P(Z < 1)/(P(Z < 2) = .8413/.9772 = 0.8609$.
(b) $P(Z < 1|Z > 0) = P(0 < Z < 1)/P(Z > 0) = .3413/.5 = 0.6826$.
(c) $P(Z > 0|Z < 1) = P(0 < Z < 1)/P(Z < 1) = .3413/.8413 = 0.4057$.

4. K 2.2, # 13: If $P(A) = .68$, $P(B) = .53$, and $P(A \cup B) = .75$, find $P(B|A)$.

Solution: Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = .68 + .53 - .75 = .46 .$$

Hence $P(B|A) = P(A \cap B)/P(A) = .46/.68 = 0.6764706$.

5. K 1.4, # 2: A certain space agency makes a sequence of attempts to launch shuttles. Suppose that the probability of a successful launch is .85, and that the sequence of launch attempts can be modeled by a Bernoulli process. Find the probability that:
(a) There are at least eight successes in the first 10 attempts.
(b) The first successful launch does not occur within the next three attempts.
(c) There are two or more failures between the third and fourth successful launches.
(d) There are six successes in the first 10 attempts, but the first two of the ten are failures.

Solution: (a) Let $T_n \equiv$ the total number of successes in the first n attempts. Then $T_n \sim \mathit{Binomial}(n, p = .85)$, and hence

$$\begin{aligned} P(T_{10} \geq 8) &= \binom{10}{8}(.85)^8(.15)^2 + \binom{10}{9}(.85)^9(.15)^1 + \binom{10}{10}(.85)^{10}(.15)^0 \\ &= 0.2758967 + 0.3474254 + 0.1968744 \approx .8202\,. \end{aligned}$$

(b) Let $Y_1 \equiv W_1$ be the number of launches until the first successful launch. Then

$$P(W_1 > 3) = P(T_3 < 1) = P(T_3 = 0) = \binom{3}{0}(.85)^0(.15)^3 = 0.003375\,.$$

(c) If Y_4 is the number of launch attempts between the 3rd and 4th successful launches, then

$$P(Y_4 \geq 3) = P(Y_4 > 2) = q^2 = (.15)^2 = 0.0225\,.$$

(d)

$$\begin{aligned} &P(\text{six successes in first 10 attempts, first 2 attempts failures}) \\ &= (.15)^2 P(\text{six successes in the next 8 attempts}) \\ &= (.15)^2 \binom{8}{6}(.85)^6(.15)^2 \\ &= 0.005346\,. \end{aligned}$$

6. K 1.4, # 5: Tests of a certain brand of car show that its breakdown rate is one breakdown in 48 months. A salesperson tells you that, therefore, you can expect to go 4 years without a breakdown.
(a) What is the actual probability that you will go 48 months without a breakdown? Assume that successive months represent Bernoulli trials, with a "success" being a breakdown in that month, so the success probability is 1/48.
(b) What is the probability that the first breakdown comes in the first year?
(c) What is the probability that you will have no breakdowns in the first year, but one or more breakdowns in the second year?

Solution: (a) Let $p = 1/48$, and let $Y_1 = W_1 \equiv$ the number of months until the first breakdown. Then

$$P(W_1 > 48) = (1-p)^{48} = (47/48)^{48} = 0.3640137\,.$$

(b) $P(W_1 \le 12) = 1 - P(W_1 > 12) = 1 - (1-p)^{12} = 0.2232532\,.$
(c) Since the number of breakdowns in the second year, $T_{24} - T_{12}$, has the same distribution as T_{12},

$$\begin{aligned}
&P(\text{no breakdowns 1st year, one or more breakdowns in 2nd year})\\
&= P(T_{12}=0)P(T_{24}-T_{12} \ge 1)\\
&= P(T_{12}=0)(1-P(T_{12}=0))\\
&= \binom{12}{0}(1/48)^0(47/48)^{12}(1-\binom{12}{0}(1/48)^0(47/48)^{12})\\
&= 0.7767468(1-0.7767468) = 0.1734112\,.
\end{aligned}$$

7. K 1.4, # 3: For a Poisson process process with an arrival rate of $\lambda = 3.5$ arrivals per minute, find the probabilities of the following events:
(a) Exactly three arrivals in the first 2 minutes.
(b) No arivals in the first 45 seconds.
(c) The fifth arrival comes in the interval between 40 seconds and 50 seconds after the fourth.

Solution: (a) $\lambda t = 3.5 \cdot 2 = 7$, so

$$P(N(2)=3) = \exp(-7)\frac{7^3}{3!} = 0.05212925\,.$$

(b) $\lambda t = 3.5 * (3/4) = 2.625$, so $P(N(3/4)=0) = \exp(-2.624) = 0.0724$.
(c) The event that the 5th arrival comes between 4/6 and 5/6 minutes after the fourth arrival is exactly the event $[4/6 < Y_5 \le 5/6]$. Since $Y \equiv Y_5 \sim$Exponential(λ),

$$\begin{aligned}
P(4/6 \le Y_5 \le 5/6) &= F_Y(5/6) - F_Y(4/6)\\
&= 1 - \exp(-3.5(5/6)) - (1 - \exp(-3.5(4/6)))\\
&= 0.0428582\,.
\end{aligned}$$

8. Bonus Problem: Look at the Negative Binomial Experiment at the Virtual Laboratories website:

`http://www.math.uah.edu/stat/bernoulli/index.html` .

(a) For $k = 2$ and $p = .4$, verify the probabilities you see plotted in blue for $P(Y=2)$, $P(Y=3)$ and $P(Y=4)$. (The random variable Y in the Experiment is W_2, the waiting time until the 2nd success in Bernoulli trials with success probability $p = .4$, in my notation.)
(b) Change the "Stop Freq 10" to "Stop Freq 100", and run the experiment. (This draws 100 samples from the Negative Binomial(2,.4) distribution. Record the sample mean "Data Mean" and standard deviation "SD Data" shown, call them $\overline{Y}_{100}$ and

S_{100}, and compute $\overline{Y}_{100} \pm 1.96 S_{100}/\sqrt{100}$. Does this interval include the population mean $\mu_Y = E(Y) = 5.0$? 0.192

Solution: (a) $W_2 \sim NegBin(2, p)$, so

$$P(W_2 = k) = \binom{k-1}{1} q^{k-2} p^2 \qquad k = 2, 3, \ldots.$$

When $p = .4$,
$P(W_2 = 2) = \binom{1}{1} q^0 p^2 = (.4)^2 = .16$
$P(W_2 = 3) = \binom{2}{1} q^1 p^2 = 2 \cdot (.6)(.4)^2 = 0.192$,
$P(W_2 = 4) = \binom{3}{1} q^2 p^2 = 3 \cdot (.6)^2 (.4)^2 = 0.173$. These do indeed agree with the picture shown at the Virtual Laboratories web site.
(b) When I did the experiment, I observed $\overline{Y}_{100} = 5.55$ and $0.61152 S_{100} = 3.12$. Hence

$$\overline{Y}_{100} \pm 1.96 S_{100}/\sqrt{100} = 5.55 \pm 1.963.12/10 = 5.55 \pm 0.61152 = (4.93848, 6.16152)\,,$$

and this interval does include the true population mean $E(W_2) = 2/p = 5$.