Statistics 394, Problem Set 2 Solutions

Wellner; 1/19/2000

1. K, 1.2, # 13.

   Solution: For $n = 6$ trials, $T \equiv T_6$, the total number of "sixes" rolled, has a Binomial$(6, 1/6)$ distribution. Hence the probability of at least one "six" in six rolls is

$$P(T \geq 1) = 1 - P(T = 0) = 1 - \binom{6}{0}(1/6)^0(5/6)^6 = 1 - (5/6)^6 \approx 0.665102\,.$$

   For $n = 12$ trials, $T \equiv T_{12}$, the total number of "sixes" rolled, has a Binomial$(12, 1/6)$ distribution. Hence the probability of at least two "sixes" in twelve rolls is

$$\begin{aligned} P(T \geq 2) &= 1 - P(T = 0) - P(T = 1) \\ &= 1 - \binom{12}{0}(1/6)^0(5/6)^{12} - \binom{12}{1}(1/6)^1(5/6)^{11} \\ &= 1 - (5/6)^{12} - (12/6)(5/6)^{11} \approx 0.6186674\,. \end{aligned}$$

   For $n = 18$ trials, $T \equiv T_{18}$, the total number of "sixes" rolled, has a Binomial$(18, 1/6)$ distribution. Hence the probability of at least three "sixes" in eighteen rolls is

$$\begin{aligned} P(T \geq 3) &= 1 - P(T = 0) - P(T = 1) - P(T = 2) \\ &= 1 - \binom{18}{0}(1/6)^0(5/6)^{18} - \binom{18}{1}(1/6)^1(5/6)^{17} - \binom{18}{2}(1/6)^2(5/6)^{16} \\ &= 1 - (5/6)^{18} - (18/6)(5/6)^{17} - (18 \cdot 17)/(2 \cdot 6^2)(5/6)^{16} \\ &\approx 0.5973457\,. \end{aligned}$$

2. K, 2.1, # 7.

   Solution: Let M denote the set of males, and L denote the set of left-handed people. Then we know that $P(M) = .5$, $P(L) = .1$, and $P(M \cap L) = .06$. Thus we compute

$$.5 = P(M) = P(M \cap L) + P(M \cap L^c) = .06 + P(M \cap L^c)\,,$$

   so $P(M \cap L^c) = .44$. Also

$$.1 = P(L) = P(L \cap M) + P(L \cap M^c) = .06 + P(L \cap M^c)\,,$$

   so $P(L \cap M^c) = .04$. This leaves just one entry in the table to calculate, namely $P(L^c \cap M^c)$. But

$$.5 = P(M^c) = P(M^c \cap L) + P(M^c \cap L^c) = .04 + P(M^c \cap L^c)\,,$$

   and hence $P(M^c \cap L^c) = .5 - .04 = .46$. This is best visualized via the following table of probabilities:

	M	M^c	total
L	.06	.04	.1
L^c	.44	.46	.90
total	.5	.5	1

3. K, 2.1, # 10.

   Solution: Let $A = [Z \le 2]$ and $B = [Z > 1]$. Then:
   (a) $P(A) = P(Z \le 2) = .9772$.
   $P(B) = P(Z > 1) = 1 - P(Z \le 1) = 1 - .8413 = .1587$.
   $P(A \cap B) = P(1 < Z \le 2) = P(Z \le 2) - P(Z \le 1) = .9772 - .8413 = .1359$.
   (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = .9772 + .1587 - .1359 = 1.0$.
   (c) The result of (b) makes sense, since $P(A \cup B) = P([Z \le 2] \cup [Z > 1]) = P(-\infty < Z < \infty) = 1$.

4. K, 2.1, # 15.

   Solution: (a) Consider G_1 and G_2 as being red balls, and the other 6 candidates being the white balls in an urn containing 8 balls. Then the probability that neither G_1 nor G_2 is chosen is the same as the probability of drawing no "red" balls in a sample of 4 balls drawn from the urn (without replacement). This probability is exactly

   $$\frac{\binom{2}{0}\binom{6}{4}}{\binom{8}{4}} = \frac{15}{70}.$$

   (b) This problem is worded a bit vaguely, but I will interpret it as asking for the probability that in four drawn for jobs in a list of eight candidates we draw G_3 and the remaining three from candidates G_4, G_5, G_6, G_7, G_8. This probability is

   $$\frac{\binom{2}{0}\binom{1}{1}\binom{5}{3}}{\binom{8}{4}} = \frac{5 \cdot 4/2}{70} = \frac{10}{70}.$$

5. A coin of radius r is thrown onto the unit square $[0,1] \times [0,1]$ in such a way that its center is uniformly distributed over the unit square. Find the probability that the coin intersects the edge (or boundary) of the square as a function of r. (See the "Buffon coin experiment" at the *Virtual Laboratory for Probability and Statistics* at

   `http://www.math.uah.edu/stat/prob/index.html` .)

   Solution: Note that the coin intersects the boundary of the square if its center lands within distance r of the boundary. It is easily seen that it fails to intersect the boundary exactly if its center falls within a smaller square with corners located at $(r,r), (r,1-r), (1-r,r), (1-r,1-r)$ for $0 < r < 1/2$. The area of this smaller square is $(1-2r)^2$, and hence the probability of the coin intersecting the boundary is $1-(1-2r)^2$ for $0 < r < 1/2$. When $r = 1/2$, the probability is 1, and when $r = 0$, the probability is 0. This probability is 1/2 when $r = (1 - 1/\sqrt{2})/2 \approx .146446$. (Compare with the blue distribution obtained at the above web site when $r = .15$

6. Two dice are thrown n times in succession. Compute the probability that double 6 appears at least once. How large does n need to be to make this probability at least 1/2?

Solution: With "success" defined as a double six, we have probability 1/36 of "success" on each trial of the basic experiment (consisting of rolling the two dice once). The total number of "successes" in n repetitions of the basic experiment has a Binomial$(n, 1/36)$ distribution:

$$P(T = k) = \binom{n}{k} \left(\frac{1}{36}\right)^k \left(1 - \frac{1}{36}\right)^{n-k}, \quad k = 0, \ldots, n\,.$$

Thus the probability of getting a double six (or a "success") at least once is

$$\begin{aligned} P(T \geq 1) &= 1 - P(T = 0) \\ &= 1 - \binom{n}{0} \left(\frac{1}{36}\right)^0 \left(1 - \frac{1}{36}\right)^n \\ &= 1 - \left(1 - \frac{1}{36}\right)^n\,. \end{aligned}$$

This probability is $\geq 1/2$ if n satisfies

$$1/2 \geq \left(1 - \frac{1}{36}\right)^n\,;$$

or if

$$\log(1/2) \geq n \log(35/36)\,;$$

or if $n \geq \log(1/2)/\log(35/36) = \log(2)/\log(36/33) \approx 24.6$.

7. The following data were given in a study of a group of 1000 subscribers to a certain magazine: in reference to sex, marital status, and eductation there were 312 males, 470 married persons, 525 college graduates, 42 male college graduates, 147 married college graduates, 86 married males, and 25 married male college graduates. Show that the numbers reported in the study must be incorrect.
Hint: Let M, W, and G denote, respectively, the set of males, married persons, and college graduates. Assume that one of the 1000 persons is chosen at random, and use (2.1.3) on Kelly page 86 to show that if the above numbers are correct, then $P(M \cup W \cup G) > 1$.

Solution: From the given numbers we conclude that $P(M) = .312$, $P(W) = .470$, $P(G) = .525$, $P(M \cap G) = .042$, $P(M \cap W) = .086$, $P(W \cap G) = .147$, and $P(M \cap W \cap G) = .025$. Hence it follows from the inclusion - exclusion formula (2.1.3) that

$$\begin{aligned} P(M \cup W \cup G) &= P(M) + P(W) + P(G) - P(M \cap W) - P(M \cap G) - P(W \cap G) \\ &\quad + P(M \cap W \cap G) \\ &= .312 + .470 + .525 - .042 - .086 - .147 + .025 \\ &= 1.057 > 1\,. \end{aligned}$$

Since the inclusion - exclusion formula is correct and all probabilities are ≤ 1, the numbers reported in this study must be *incorrect*.

8. **Extra bonus problem:** A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

   Solution: If the 12 blocks were numbered 1 through 12, then the number of arrangements would be 12!. When we account for the 6! ways of arranging the 6 black balls and the 4! ways of arranging the red balls, we easily see that the total number of arrangements of 12 blocks, 6 of which are black, and 4 of which are red, 1 of which is white, and 1 of which is blue, is

$$\frac{12!}{6!4!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 11 \cdot 5 \cdot 9 \cdot 8 \cdot 7 = 27,720\,.$$