

**Handout 9: Math/Stat 394: Probability I**  
**Expectation Summary**  
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**Definition 1.** Random outcome  $X, Y$  yields payoff  $g(X, Y)$  with average value

$$Eg(X, Y) \equiv \begin{cases} \sum \sum_{all\ x, y} g(x, y) p_{X, Y}(x, y) & \text{(discrete rv's)} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X, Y}(x, y) dx dy & \text{(continuous rv's)}. \end{cases}$$

**Example 1.**

$\mu_X \equiv EX$  and  $\sigma_X^2 \equiv Var(X) \equiv E(X - EX)^2$ .

$\mu_Y \equiv EY$  and  $\sigma_Y^2 \equiv Var(Y) \equiv E(Y - EY)^2$ .

$\mu_X$  and  $\sigma_X$  measure “center” and “spread” of  $X$ ;

$\mu_Y$  and  $\sigma_Y$  measure “center” and “spread” of  $Y$ ;

$\sigma_{X, Y} \equiv Cov(X, Y) \equiv E[(X - EX)(Y - EY)]$

and  $\rho_{X, Y} \equiv Corr(X, Y) \equiv \sigma_{X, Y} / (\sigma_X \sigma_Y)$

measure the relationship between  $X$  and  $Y$ .

**Theorem 1.**

$$E(a) = a$$

$$E(aX + b) = aE(X) + b$$

$$E(aX + bY + c) = aE(X) + bE(Y) + c$$

$$E(X + Y) = E(X) + E(Y)$$

$$\text{or } \mu_{aX+b} = a\mu_X + b$$

$$\text{or } \mu_{aX+bY+c} = a\mu_X + b\mu_Y + c \text{ always}$$

$$\text{or } \mu_{X+Y} = \mu_X + \mu_Y \text{ always}$$

$$Var(aX + b) = a^2 Var(X)$$

$$Var(X) = 0 \text{ iff } P(X = \mu_X) = 1$$

$$Var(X + Y) = Var(X) + Var(Y)$$

if  $X$  and  $Y$  are independent

$$Var(X + Y)$$

$$= Var(X) + Var(Y) + 2Cov(X, Y)$$

always

$$Var(aX + bY + c)$$

$$= a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$$

$$\text{or } \sigma_{aX+b} = |a|\sigma_X$$

$$\text{or } \sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2$$

if  $X$  and  $Y$  are independent

$$\text{or } \sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\sigma_{X,Y}$$

always

$$\text{or } \sigma_{aX+bY+c}^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{X,Y}$$

Computing Formulas:

$$Var(X) = E(X^2) - (EX)^2$$

$$Cov(X, Y) = E(XY) - EX \cdot EY$$

$$Var(X) = Cov(X, X)$$

$$Cov(X, Y) = 0 \text{ whenever } X \text{ and } Y$$

are independent

The **key fact** in all this:

$E\{g(X)h(Y)\} = E\{g(X)\} \cdot E\{h(Y)\}$  for any  $g(\cdot)$  and  $h(\cdot)$  whenever  $X$  and  $Y$  are independent.

Thus  $Cov(X, Y) = E\{(X - \mu_X)(Y - \mu_Y)\} = E(X - \mu_X) \cdot E(Y - \mu_Y) = 0 \cdot 0 = 0$  when  $X$  and  $Y$  are independent.

$$Cov\left[\sum_{i=1}^m a_i X_i, \sum_{j=1}^n b_j Y_j\right] = \sum_{i=1}^m \sum_{j=1}^n a_i b_j Cov[X_i, Y_j]$$

$$Var\left[\sum_{i=1}^n a_i X_i\right] = \sum_{i=1}^n a_i^2 Var[X_i] + \sum_{i \neq j} a_i a_j Cov[X_i, X_j]$$

$$Var\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n Var[X_i] + \sum_{i \neq j} Cov[X_i, X_j]$$

$$= nVar(X_1) + n(n-1)Cov[X_1, X_2]$$

if all variances and covariances are equal.

$$\begin{aligned} \text{Var}\left[\sum_{i=1}^n X_i\right] &= \sum_{i=1}^n \text{Var}(X_i) && \text{if the } X_i\text{'s are independent} \\ &= n\text{Var}(X_1) && \text{if all the variances are equal.} \end{aligned}$$

**Example 2.** (Urn mean and variance). An urn contains  $N$  balls numbered  $a_1, \dots, a_N$ . Let  $X$  denote the  $a$ -value of one ball drawn at random from the urn. Then

$$\mu_X = \bar{a}_N \equiv \frac{1}{N} \sum_{i=1}^N a_i$$

and

$$\sigma_X^2 = \sigma_a^2 \equiv \frac{1}{N} \sum_{i=1}^N (a_i - \bar{a}_N)^2 = \frac{1}{N} \sum_{i=1}^N a_i^2 - \bar{a}_N^2.$$

**Example 3.** (Sampling *with* replacement from an urn.) Let  $T_n \equiv X_1 + \dots + X_n$ . Then

$$E(T_n) = n\bar{a}_N \quad \text{and} \quad \text{Var}(T_n) = n\sigma_a^2,$$

and hence for  $\bar{X}_n \equiv T_n/n$ ,

$$E(\bar{X}_n) = \bar{a}_N \quad \text{and} \quad \text{Var}(\bar{X}_n) = \frac{1}{n}\sigma_a^2.$$

Note that  $\text{Cov}(X_1, X_2) = 0$  since the  $X_i$ 's are independent.

**Example 4.** (Sampling *without* replacement from an urn.) Let  $T_n \equiv X_1 + \dots + X_n$ . Then

$$E(T_n) = n\bar{a}_N \quad \text{and} \quad \text{Var}(T_n) = n\sigma_a^2 \left(1 - \frac{n-1}{N-1}\right),$$

and hence for  $\bar{X}_n \equiv T_n/n$ ,

$$E(\bar{X}_n) = \bar{a}_N \quad \text{and} \quad \text{Var}(\bar{X}_n) = \frac{1}{n}\sigma_a^2 \left(1 - \frac{n-1}{N-1}\right).$$

Here  $\text{Cov}(X_1, X_2) = -\sigma_a^2/(N-1)$ . The “correction factor”  $(1 - (n-1)/(N-1))$  is due to the covariance no longer being 0. It has the same value no matter what numbers  $a_1, \dots, a_N$  are in the urn.

**Proof.**

$$0 = \text{Var}(X_1 + \cdots + X_N) = N\sigma_a^2 + N(N-1)\text{Cov}(X_1, X_2)$$

so  $\text{Cov}(X_1, X_2) = -\sigma_a^2/(N-1)$ . Then

$$\begin{aligned} \text{Var}(X_1 + \cdots + X_n) &= n\sigma_a^2 + n(n-1)\text{Cov}(X_1, X_2) \\ &= n\sigma_a^2 + n(n-1) \left( -\frac{\sigma_a^2}{N-1} \right) \\ &= n\sigma_a^2 \left( 1 - \frac{n-1}{N-1} \right). \end{aligned}$$

**Example 5.** (Variance of Hypergeometric). If the numbers on the balls are  $a_1 = \dots = a_R = 1$ ,  $a_{R+1} = \dots = a_N = 0$ , then  $T_n = X_1 + \cdots + X_n$  is the number of red balls in the sample when the urn contains  $R$  red balls. Thus  $T_n \sim \text{Hypergeometric}(R, N, n)$ . The results of Example 4 tell us in this special case that

$$E(T_n) = n\bar{a}_N = n\frac{R}{N},$$

and

$$\text{Var}(T_n) = \left( 1 - \frac{n-1}{N-1} \right) n\sigma_a^2 = \left( 1 - \frac{n-1}{N-1} \right) n(R/N)(1-R/N)$$

since

$$\sigma_a^2 = \frac{1}{N} \sum_{i=1}^N a_i^2 - \bar{a}_N^2 = R/N - (R/N)^2 = (R/N)(1-R/N).$$

Note that these expressions differ only by the finite sampling correction factor  $(1 - (n-1)/(N-1))$  from the comparable expressions for the mean and variance of  $T_n$  if the sampling is carried out with replacement: in that case  $T_n \sim \text{Binomial}(n, p = R/N)$  and

$$E(T_n) = np = nR/N, \quad \text{Var}(T_n) = np(1-p) = n(R/N)(1-R/N).$$