MATH/STAT 394: Probability I
Wellner, 2/2/2000

Practice Midterm Exam: SOLUTIONS

1. Adult brain weights are normally distributed with a mean weight μ of 1400 grams and a standard deviation σ of 110 grams. What fraction of a group of randomly chosen brain weights would fall between 1620 grams and 1290 grams?

   Solution: Let $X \sim N(1400, 110^2)$ be a randomly chosen brain weight. Then $Z \equiv (X - 1400)/110 \sim N(0, 1)$ and hence

$$
\begin{aligned}
P(1290 < X < 1620) &= P\left(\frac{1290 - 1400}{110} < \frac{X - 1400}{110} < \frac{1620 - 1400}{110}\right) \\
&= P(-1 < Z < 2) = P(Z < 2) - P(Z < -1) \\
&= .9772 - .1587 = .8185\,.
\end{aligned}
$$

2. Suppose an urn contains 8 red balls and 12 blue balls. We sample 5 of them from the urn without replacement.
   (a) What is the probability the five outcomes are $RBRRB$?
   (b) What is the probability that exactly 2 of the 5 sampled would be red?
   (c) What is the name of the distribution that is relevant in (b)?

   Solution: (a) $(8/20)(12/19)(7/18)(6/17)(11/16) = 0.023839\,.$
   (b) $\binom{8}{2}\binom{12}{3}/\binom{20}{5} = 0.3973.$
   (c) Hypergeometric$(R = 8, N = 20, n = 5)$.

3. Demands for service at a business facility occur (throughout the period of interest) according to a Poisson process with an intensity of 5 per minute.
   (a) Find the probability of exactly 2 demands in the next 1 minute.

(b) Find the probability of at least 40 demands in the next 10 minutes.
(c) Find the probability that we wait more than 36 seconds (= 3/5 minute) before getting 2 demands.

Solution: Here $\nu = 5$ demands per minute.
(a) $\mathbb{N}(1) \sim Poisson(\lambda = \nu t = 5 \cdot 1 = 5)$, so

$$P(\mathbb{N}(1) = 2) = e^{-5}\frac{5^2}{2!} = 0.0842\,.$$

(b) $\mathbb{N}(10) \sim Poisson(\lambda = \nu t = 5 \cdot 10 = 50)$, which is well-approximated by a Normal distribution with the same mean and variance, $N(50, 50)$. Hence, with $Z \sim N(0,1)$,

$$\begin{aligned} P(\mathbb{N}(10) \geq 40) &= P(\mathbb{N}(10) \geq 39.5) \\ &= P\left(\frac{\mathbb{N}(10) - 50}{\sqrt{50}} \geq \frac{39.5 - 50}{\sqrt{50}}\right) \\ &\doteq P(Z \geq -1.485) = P(Z \leq 1.485) = .9312\,. \end{aligned}$$

(c) By the fundamental event identity for the Poisson process, $[W_2 > 3/5] = [\mathbb{N}(3/5) < 2]$ where $\mathbb{N}(3/5) \sim$ Poisson$(\lambda = \nu t = 5(3/5) = 3)$. Hence it follows that

$$P(W_2 > 3/5) = P(\mathbb{N}(3/5) < 2) = e^{-3} + e^{-3}\frac{3^1}{1!} = e^{-3}(1+3) = 0.1991\,.$$

4. The U.S. Senate consists of 2 senators from each of 50 states, making a total of 100 senators.
(a) In how many ways can a subgroup of 4 states be chosen?
(b) A committee of size 4 is to be formed, under rules that prohibit any 2 senators from the same state. According to the rules, how many committee choices are possible?

Solution: (a) $\binom{50}{4} = 50 \cdot 49 \cdot 48 \cdot 47/(4 \cdot 3 \cdot 2 \cdot 1) = 230300\,.$
(b) We can choose a committee by first choosing 4 states, and then choosing one of the two senators from each of the 4 states. Thus the total number of such committees is $\binom{50}{4} \cdot 2^4 = 230300 \cdot 16 = 3684800\,.$

5. In the first two columns of the following table, we have provided a percentage distribution for the religious affiliation of the voters in a hypothetical city. The data are based on the national distribution for religious preference found in *Emerging Trends*, published by the Princeton Religion Research Center of Princeton, New Jersey. In the third column of the table, we show the percentage of Democrats in each religious group of voters. The table show, for instance, that 28% of the voters in the city are Catholic, and that 53% of the catholic voters in the city are Democrats.

Religion	Percentage of voters	Percentage Democrats
Catholic	28	53
Jewish	2	61
Prostestant	57	42
Other	4	58
None	9	67
	100	

(a) What percentage of the voters are Democrats?
(b) What percentage of the Democrats are Protestant?

Solution: Let C, J, P, O, N be the events that a randomly chosen voter is Catholic, Jewish, Protestant, Other, or None (in religious preference). Let D be the event that a randomly chosen voter is a Democrat.

(a) The first column gives $P(C) = .28$, $P(J) = .02$, $P(P) = .57$, $P(O) = .04$, and $P(N) = .09$. The second column gives $P(D|C) = .53$, $P(D|J) = .61$, $P(D|P) = .42$, $P(D|O) = .58$, and $P(D|N) = .67$. Thus

$$\begin{aligned} P(D) &= P(DC) + P(DJ) + P(DP) + P(DO) + P(DN) \\ &= P(D|C)P(C) + P(D|J)P(J) + P(D|P)P(P) \\ &\quad + P(D|O)P(O) + P(D|N)P(N) \\ &= .28(.53) + .02(.61) + .57(.42) + .04(.58) + .09(.67) \\ &= .4835\,. \end{aligned}$$

(b) Now

$$P(P|D) = \frac{P(P \cap D)}{P(D)} = \frac{P(D|P)P(P)}{P(D)} = \frac{.57(.42)}{.4835} = 0.4951\,.$$