MATH/STAT 394: Probability I

Wellner, 2/8/2000

Midterm Exam Solutions

1. Assume that the number of miles a driver gets on a set of radial tires is normally distributed with a mean of $30,000$ miles and a standard deviation of 5000 miles. What proportion of drivers will get at least $22,000$ miles?

   Solution: Let's measure mileage in thousands of miles. Letting $X \equiv$mileage on these tires for a randomly chosen driver, we are assuming that $X \sim N(30,5)$. Then

$$\begin{aligned} P(X \geq 22) &= P\left(\frac{X-30}{5} \geq \frac{22-30}{5}\right) \\ &= P(Z \geq -1.6) = P(Z \leq 1.6) = .9452\,. \end{aligned}$$

2. Suppose an urn contains 7 white balls and 18 black balls. We sample 6 of them from the urn without replacement.
   (a) What is the probability the six outcomes are $WBBBWW$?
   (b) What is the probability that exactly 3 of the 6 sampled would be white?
   (c) What is the name of the distribution that is relevant in (b)?
   (d) If we sampled *with* replacement, what is the probability that exactly 3 of the 6 sampled would be white?

   Solution: (a) $(7/25)(18/24)(17/23)(16/22)(6/21)(5/20) = 0.0081$.
   (b) $\binom{7}{3}\binom{18}{3}/\binom{25}{6} = 0.1613$.
   (c) Hypergeometric($R = 7, N = 25, n = 6$).
   (d) $\binom{6}{3}(7/25)^3(18/25)^3 = 0.1639$.

3. People enter a gambling casino at a rate of 1 for every 2 minutes; i.e. the intensity is $\nu = .5$ per minute. Suppose that the arrival process is a Poisson process with this intensity.
(a) What is the probability that no one enters between 12:00 and 12:05?
(b) What is the probability that at least 4 people enter between 12:00 and 12:05?
(c) What is the probability that we wait more than 5 minutes for the third person to enter?

Solution: (a) Since $\mathbb{N}(5) \sim Poisson(\lambda = .5(5) = 2.5)$,

$$P(\mathbb{N}(5) = 0) = e^{-2.5} = 0.0821\,.$$

(b) Since $\mathbb{N}(5) \sim Poisson(2.5)$ as in (a),

$$\begin{aligned} P(\mathbb{N}(5) \geq 4) &= 1 - P(\mathbb{N}(5) \leq 3) \\ &= 1 - e^{-2.5}\{1 + 2.5 + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!}\} \\ &= 0.2424\,. \end{aligned}$$

(c) By using the fundamental identity for the Poisson process, we compute

$$P(W_3 > 5) = P(\mathbb{N}(5) < 3) = e^{-2.5}\{1 + 2.5 + \frac{(2.5)^2}{2!}\} = 0.54381\,.$$

4. (a) How many different 7-place license plates are possible if the first 2 places are for letters, and other 5 for numbers?
(b) How many different 7-place license plates are possible if we insist that no letter or number can be repeated in a single license plate?

Solution: (a) $26^2 \cdot 10^5 = 67600000$.
(b) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 19656000$. (The ratio of these two numbers is 3.439; there are about 3.4 times as many possible license plates if repetitions of letters and numbers are allowed.)

5. At a psychiatric clinic the social workers are so busy that on the average, only 60% of potential new patients that telephone are able to talk

immediately with a social worker when they call. The other 40% are asked to leave their phone numbers. About 75 percent of the time a social worker is able to return the call on the same day, and the other 25 percent of the time the caller is contacted on the following day. Experience at the clinic indicates that the probability a caller will actually visit the clinic for consultation is .8 if the caller was immediately able to speak to a social worker, whereas it is .6 and .4, respectively, if the patient's call was returned the same day or the following day.
(a) What percentage of people that telephone, visit the clinic for consultation?
(b) What percentage of patients that visit the clinic had their telephone calls answered immediately?

Solution: (a) Let V = the event that a caller visits the clinic for consultation, I = the event that a caller immediately speaks to a social worker, S = the event that a caller is contacted later on the same day, and F = the event that a caller is contacted on the following day. We are given $P(I) = .6$, $P(S) = .3$, $P(F) = .1$, and $P(V|I) = .8$, $P(V|S) = .6$, $P(V|F) = .4$. Thus

$$\begin{aligned} P(V) &= P(VI) + P(VS) + P(VF) \\ &= P(V|I)P(I) + P(V|S)P(S) + P(V|F)P(F) \\ &= .8 \cdot .6 + .6 \cdot .3 + .4 \cdot .1 \\ &= .7\,. \end{aligned}$$

(b) $P(I|V) = P(IV)/P(V) = P(V|I)P(I)/P(V) = .8 \cdot .6/.7 = 0.6857\,.$
(This is Bayes formula!)