

MATH/STAT 394: Probability I

Wellner, 3/14/2000

Final Exam

Instructions: You may consult your textbook or your notes during this exam. Use a separate sheet of paper, write your name on the top left corner of the sheet, and **write as legibly as possible**. You should try to answer all 9 questions. You must work completely on your own. The total number of points is 240.

- (32 points). Suppose that $R \sim \text{Poisson}(\mu)$, $S \sim \text{Poisson}(\nu)$, and $T \sim \text{Poisson}(\tau)$ are all independent. Suppose that $X \equiv R + T$, $Y \equiv S + T$, $Z \equiv R - T$.
 - Compute $E(X)$, $Var(X)$, $E(Y)$, $Var(Y)$.
 - Compute $Cov(X, Y)$ and $\rho_{X, Y}$, the correlation between X and Y .
 - Compute $Cov(X, Z)$ assuming that $\mu = \tau$. Are X and Z independent? (Justify your answer.)
 - What are the marginal distributions of X and Y ?
- (20 points) Use the normal approximation to the binomial distribution to find the approximate probability that in 99 tosses of a fair coin there are fewer than 40 heads. Use a continuity correction.
- (20 points) The event whose probability is found in the preceding problem, “fewer than 40 heads in 99 tosses” can be restated as “100 or more trials are needed to produce the 40th head”; i.e. in the notation of our section on the Bernoulli process, $[T_{99} < 40] = [W_{40} > 99]$. You can approximate the probability of the event on the right side by using the Central Limit Theorem. Do this and compare with the result of the previous problem. Which of the two approximations is likely to be more accurate? (Justify/explain your answer.)
- (32 points) Suppose that the marginal distribution of X is $\text{Gamma}(2, \nu)$, and the conditional distribution of Y given $X = x$ is $\text{Uniform}(-x, x)$.
 - Determine the joint density/mass function of X and Y . Show where the joint density is positive in a diagram.
 - Determine the marginal density of Y . [Hint: consider positive and negative values of y , the argument of the marginal density, separately.]
 - Are X and Y independent? Justify your answer.
 - Determine the conditional density of X given $Y = y$.

5. (32 points) Customers arrive at the check-out area of an Eagle Hardware store at a rate of 4 per minute on Saturdays in the spring. Break the upcoming hours into 12 disjoint five minute intervals. “Customer overload” is said to occur if such a five minute interval sees the arrival of at least 26 customers.
- A. What is the probability of:
- Customer overload in the first five minute interval?
 - The first hour (i.e. sometime in the first 12 five minutes intervals)?
 - The 2nd customer overload occurring in the 8th interval?
- B. Evaluate the mean and variance of the number of intervals until the 6th customer overload.
6. (32 points). Roll a pair of dice (one red and one white) with outcomes X_1 and X_2 . Let $T = X_1 + X_2$, $D = X_1 - X_2$, and $M = \max\{X_1, X_2\}$. Evaluate:
- $P(D \geq 1 | X_1 + X_2 \leq 8)$.
 - $P(M \leq 4 | T \geq 6)$.
 - $P(|D| \leq 1 | M \geq 5)$.
7. (24 points) Joe and Harry take turns rolling two dice with Joe going first. Joe wins if he rolls “6” or “8”, while Harry wins if he rolls “4” or “7”. Determine $P(\text{Joe wins})$ in this sequential contest.
8. (24 points) Consider drawing (without replacement) from an urn consisting of 100 balls, 20 of which are labeled with the number 1, 30 of which are labeled with the number 5, 30 of which are labeled with the number 10, and 20 of which are labeled with the number 15. Determine the mean and standard deviation of:
- One randomly chosen ball.
 - The sample mean of a sample of 40 randomly chosen (without replacement) balls.
 - Use a central limit theorem to approximate $P(\bar{X}_n \geq 8.77)$.
9. (24 points) Evaluate (or approximate) as accurately as you can, the probability of at least 7 honor cards in at least 12 of 200 hands of bridge. (Recall that a deck of 52 cards contains 16 “honor cards”, and that a bridge hand contains 13 cards.)