4 GENERIC ANALYSIS OF UTILITY MODELS

John M. Miyamoto

Introduction

It is now firmly established that expected utility (EU) theory and subjective expected utility (SEU) theory are descriptively invalid (Kahneman and Tversky, 1979; Luce, 1988b; MacCrimmon and Larsson, 1979; Slovic and Lichtenstein, 1983; Weber and Camerer, 1987). Descriptive utility theory is undergoing extensive revision, stimulated by empirical findings that challenge existing theories, and by new theories that more adequately account for the cognitive processes that underly preference behavior (Becker and Sarin, 1987; Bell, 1982; Kahneman and Tversky, 1979; Loomes and Sugden, 1982; Luce, 1988a, 1990; Luce and Narens, 1985; Quiggin, 1982). Although these developments are undoubtedly salutory for the theory and practice of decision making, it might appear that in the short term they undermine the usefulness of multiattribute utility theory (MAUT), or at least that part of MAUT that is built upon EU or SEU assumptions. (Henceforth, I will refer only to SEU theory, noting that EU theory can be construed as a special case of SEU theory.) A substantial part of MAUT methodology is based on preference assumptions that characterize classes of utility models under the assump-

W. Edwards (ed.), UTILITY THEORIES: MEASUREMENTS AND APPLICATIONS.

tion that SEU theory is valid (Keeney and Raiffa, 1976; von Winterfeldt and Edwards, 1986). The strong evidence against the descriptive validity of SEU theory might appear to undermine or even invalidate those parts of MAUT methodology that assume SEU theory in deriving implications from patterns of preference. A major goal of this chapter is to show that this is in fact not the case.

Of course, not all methods of MAUT analysis are based on SEU theory. For example, formalizations of MAUT models in terms of preferences under certainty or strengths of preference do not require SEU assumptions (Dyer and Sarin, 1979; Keeney and Raiffa, 1976; Krantz, Luce, Suppes, and Tversky, 1971; von Winterfeldt and Edwards, 1986). The present critique is only relevant to that part of MAUT methodology that is based on the assumptions of EU or SEU theory. I will refer to this as the risk-based part of MAUT. The essential feature of risk-based MAUT is that in this framework specific utility models are formalized in terms of preferences among hypothetical lotteries, rather than in terms of strengths of preference, or riskless preferences. For example, in the risk-based methodology, additive, multiplicative, and multilinear utility models are formalized by utility independence assumptions that describe how preference among gambles for particular attributes are affected by the levels of other attributes (Keeney and Raiffa, 1976). Similarly, Pratt (1964) described measures of risk aversion, defined in terms of choices between lotteries and certain outcomes, that describe the shape of utility functions. Tests of utility independence assumptions and of Pratt's characterizations of risk aversion are important in risk-based MAUT because they are used to diagnose the form of utility functions in specific domains. Such assumptions constitute necessary and sufficient conditions for specific utility models under the assumption that SEU theory is valid. Comprehensive descriptions of risk-based MAUT methodology are available in Keeney and Raiffa (1976) and von Winterfeldt and Edwards (1986); these monographs also describe methods of utility modeling that are not risk-based, that is, methods of utility modeling that do not require the assumption that SEU theory is valid.

The question addressed in this chapter is whether empirical violations of SEU theory undermine risk-based MAUT methodology. Should we cease to regard tests of utility independence and Pratt's (1964) risk characterizations as meaningful in descriptive research because they carry their implications within the framework of SEU theory? Alternatively, we might continue to incorporate risk-based MAUT into descriptive and prescriptive analyses with pious remarks concerning the approximate validity of SEU theory, while harboring a bad conscience over the unknown consequences of the divergence between SEU theory and empirical reality. What I want to show in this chapter is that there is a third alternative that is clearly preferable to either of the first two. I will propose a new theoretical framework within which risk-based MAUT methods can be justified, which does not commit one to the descriptively invalid claims of SEU theory. In other words, I will show how methods of utility modeling that were heretofore justified within the SEU framework can be reinterpreted and justified within a new utility theory that is more consistent with what we presently know about preference behavior.

The theory that I will propose is called *generic utility theory* (Miyamoto, 1988). In formalizing generic utility theory, an attempt was made to construct an axiomatic preference theory that would satisfy three desiderata:

1. The assumptions of the framework should be consistent with existing empirical studies of preference;
2. The framework should provide a basis for methods of MAUT modeling that were previously based on SEU theory; and
3. Utility analyses developed within the framework should be interpretable from a wide variety of theoretical standpoints.

The first desideratum is the obvious requirement that the theory should not be refuted by existing findings. The second reflects the desire not to lose risk-based MAUT methods as we shift from a SEU to a non-SEU framework. The third reflects the desire for a truly generic theory, that is to say, one that is consistent with many theories and has few distinctive features of its own. Because the assumptions of generic utility theory are weak, they are implied by a number of other utility theories including EU and SEU theory (Luce and Raiffa, 1957): prospect theory (Kahneman and Tversky, 1979); the dual bilinear model (Luce and Narens, 1985); Karmarkar's (1978) subjectively weighted utility (SWU) model; rank dependent utility theory (Luce, 1988a, 1990), (Quiggin, 1982), (Yaari, 1987); and Edwards' (1962) additive subjective expected utility (ASEU) and nonadditive subjective expected utility (NASEU) models[1]. Although the assumptions of generic utility theory are weak, they are sufficiently strong to constitute a logically sufficient basis from which to derive the implications of utility independence assumptions and Pratt's risk characterizations. Thus, with minor modifications to be described below, we can continue to use risk-based methods for analyzing utility models while basing the analyses on weak assumptions that are empirically more plausible than SEU theory. Generic utility theory is generic in the true sense of the word because it possesses this mixture of weakness and strength: Standard risk-based methods for analyzing utility models can be formalized under the assumptions of generic utility theory, and utility analyses that are carried out under these assumptions are interpretable from the standpoint of stronger theories that imply it.

It must be emphasized that generic utility theory is not proposed as a general theory of preference under risk, and hence, it is not a competitor of stronger theories like prospect theory, the dual bilinear model, and rank dependent utility theory. Rather, the purpose of generic utility theory is to provide a framework for utility modeling. Utility modeling, as I understand the term, is the enterprise of investigating the form of utility functions in specific domains, like the domains of health outcomes or environmental outcomes. When engaged in utility modeling, one's primary goal is to construct a mathematical model that characterizes someone's values in the given domain, rather than to test general assumptions of preference theory. A foundation for utility modeling may remain noncommittal on important issues in preference theory if empirical criteria for utility models can be formulated without resolving these issues. Working within the generic utility framework, researchers can reach agreement in the utility analysis of particular outcome domains, even while contimuing to debate fundamental issues of preference theory. Thus, generic utility theory complements stronger theories, such as SEU theory, prospect theory, rank dependent utility theory, and the dual bilinear model, by providing a framework for utility modeling that is meaningful from the standpiont of these theories, without committing one to assumptions that are idiosyncratic to one strong theory and not to others. These remarks assume that in the near future no descriptive theory will predominate to the exclusion of all competitiors, for if such a dominating theory were established, one would naturally axiomatize and test utility models within the framework of this dominant theory, and the interpretability of analyses from alternative standpoints would be irrelevant.

The remainder of this chapter consists of four sections. First, I will present generic utility theory, and discuss its axiomatic foundation. Second, I will define more carefully the class of utility theories that are consistent with generic utility theory. Prospect theory is the most complex of these cases, and I will discuss it first. It will then be clear what types of theories are consistent with generic utility theory. Third, an empirical investigation of a multiplicative utility model will be presented within the generic utility framework. This empirical study exemplifies the use of generic utility theory in utility modeling. Finally, the role of generic analyses will be discussed with respect to general issues in theory construction and utility modeling.

The Generic Utility Representation

Let C denote a set of consequences; let p denote any fixed probability such that $p \neq 0, 1$; let (x, p, y) denote a gamble with a p chance of winning x and a $1 - p$ chance of winning y; let $G(p)$ denote the set of all (x, p, y) with $x, y \in C$. $G(p)$ will be referred to as the set of p-gambles. I will not distinguish notationally between the preference ordering over outcomes and gambles. Thus, I write $x \geqslant_p y$ if outcome x is at least as preferred as outcome y, and $(w, p, x) \geqslant_p (y, p, z)$ if the gamble (w, p, x) is at least as preferred as the gamble (y, p, z). Many theories of preference under risk postulate the existence of a real-valued function, U, and positive weights, s and t, that depend on p such that $s + t = 1$ and

$$(w, p, x) \geqslant_p (y, p, z) \quad \text{iff} \quad sU(w) + tU(x) \geqslant sU(y) + tU(z), \qquad (1)$$

for every (w, p, x) and (y, p, z) in $G(p)$. The utility representation defined by condition (1) will be called *the standard model for p-gambles.* This model is implied by EU theory with the constraint that $s = p$ and $t = 1 - p$. It is also implied by SEU theory and Karmarkar's (1978) SWU theory, with somewhat different constraints on the coefficients, s and t. These utility theories are described more fully below. An axiomatization of (1) is stated in Krantz et al. (1971, chapter 6).

The generic utility representation is a generalization of the standard model for p-gambles. Let $S(p)$ be the set of all (a, p, x) such that $a, x \in C$ and $x \geqslant_p a$, and let $T(p)$ be the set of all (b, p, y) such that $b, y \in C$ and $b \geqslant_p y$. I will call $S(p)$ an *upper triangular set of p-gambles*, and $T(p)$ a *lower triangular set of p-gambles*. The terminology is motivated by the fact that if C were a set of money rewards and each (a, p, x) were assigned the coordinates (a, x) in the $C \times C$ plane, then the upper triangular set would be the set of all gambles that are on or above the main diagonal, and the lower triangular set would be the set of all gambles that are on or below the main diagonal (see figure 4–1). A set of gambles will be said to be triangular if it is either an upper of lower triangular set.

Suppose that $R(p)$ is a triangular set of p-gambles. The structure $(C, p, R(p), \geqslant_p)$ will be said to have a generic utility representation iff there exists a real-valued function, U, and real coefficients, α and β, such that $\alpha > 0$, $\beta > 0$, and

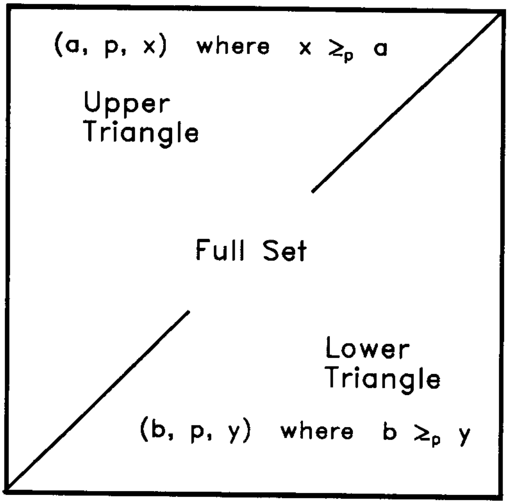

Figure 4–1. A gamble of the form, (a, p, x), is assigned the coordinates, (a, x). Gambles above the diagonal are in the upper triangular set, and gambles below the diagonal are in the lower triangular set.

$$(a, p, x) \geqslant_p (b, p, y) \quad \text{iff} \quad \alpha U(a) + \beta U(x) \geqslant \alpha U(b) + \beta U(y) \qquad (2)$$

for every $(a, p, x), (b, p, y) \in R(p)$. The coefficients, α and β, depend on the value of p, but since p is held constant in the present analysis, α and β can be treated as constants. The generic utility representation differs from the standard model for p-gambles only in that the domain of the representation is restricted to the preference order over gambles in a triangular set. To see why this is more general than the standard model, suppose that $S(p)$ is an upper triangular set of p-gambles, and $T(p)$ is a lower triangular set of p-gambles. It could be the case that $(C, p, S(p), \geqslant_p)$ satisfies the generic utility representation with respect to one pair of coefficients, α' and β', and $(C, p, T(p), \geqslant_p)$ satisfies the generic utility representation with respect to a second pair of coefficients, α'' and β'', while $\alpha' \neq \alpha''$ and $\beta' \neq \beta''$. The standard model requires that $\alpha' = \alpha'' = s$, and $\beta' = \beta'' = t$. The generic utility representation also differs from the standard model in that condition (2) places no restriction on the coefficients, α and β, other than that they are positive. Although condition (2) is consistent with theories that require that the coefficients satisfy $\alpha + \beta = 1$, it does not require that this constraint be satisfied.

An axiomatization of the generic utility representation is presented in Appendix 1 of this chapter. Ironically, the most interesting feature of the axiomatization is that it differs so little from axiomatizations of the standard utility model for p-gambles. Essentially, one takes an axiomatization of the standard model and modifies the axioms to apply only to p-gambles in a triangular set. Aside from the restriction to a triangular set of p-gambles, the main differences between the axiomatizations of the generic utility representation and the standard model is that first-order stochastic dominance is assumed (Axiom 4 of Appendix 1), which is slightly stronger than the independence assumption in the axiomatization of the standard model (Krantz et al., 1971), and an existential assumption (Axiom 10 of Appendix 1) is added to ensure that the preference order does not contain empty gaps (if $x >_p y$, then there exists z such that $x >_p z >_p y$). The representation and uniqueness theorem for the generic utility representation is stated in Appendix 1, and the proof of the theorem is sketched. Although the axiomatization of the generic utility representation is a straightforward generalization of previous work on additive models, the method used in proving the existence of the representation is new. Because the preference order in the generic theory is only defined over a triangular set of p-gambles, special techniques are required to prove the existence and interval-scale uniqueness of the utility scale (Miyamoto, 1988). From the standpoint of axiomatic measurement theory, the proof is a nontrivial extension of additive conjoint measurement. Wakker (1989a, 1989b) independently proved a utility representation theorem that includes the generic utility representation as a special case.

Next, I will show that prospect theory implies that the generic utility representation is satisfied by particular classes of gambles. This relationship is important because it establishes that studies of utility models within the generic utility framework can be interpreted from the standpoint of prospect theory. Furthermore, it implies that axiomatizations of MAUT models that were previously formalized under SEU assumptions can be incorporated into prospect theory. The discussion of prospect theory will clarify the point that any theory that implies that the generic utility representation is capable of axiomatizing MAUT models.

Prospect Theory

Here, I will only develop those aspects theory that are needed to show the relation between it and generic utility theory. The discussion will focus on the prospect theory analysis of preferences for p-gambles. Many important features of prospect theory, such as the editing or framing of gambles, the shapes of the value and probability weighting functions, and its generalization to gambles with three or more outcomes, will be omitted (see Kahneman and Tversky, 1979, 1984; Tversky and Kahneman, 1981).

Prospect theory postulates that the subjective value of an outcome is evaluated in terms of a comparison to a reference level. An outcome is categorized as a gain if it exceeds the reference level, and as a loss if it is below the reference level. For present purposes, the main reason for distinguishing gains from losses is that different rules determine the subjective value of a gamble, depending on whether the outcomes of the gamble are gains or losses. To describe these rules, suppose that $V(a, p, x)$ denotes the subjective value of (a, p, x). V is a real-valued function that preserves the preference order over p-gambles (and other gambles not discussed here). According to prospect theory, there exists a real-valued function, v, that maps outcomes to subjective values, and a function, π, that maps probabilities to subjective weights in the unit interval. For example, $v(x)$ is the subjective value of the outcome x, and $\pi(p)$ is the subjective weight of the probability p. Let r denote the reference level in the outcome domain. Prospect theory requires that v and π satisfy the constraints $v(r) = 0$, $\pi(0) = 0$, and $\pi(1) = 1$.

Figure 4–2 presents a graphical representation of the classification of p-gambles in prospect theory. Suppose that a gamble (a, p, x) is assigned the coordinates (a, x) in figure 4–2, where the horizontal axis represents an ordering of the first outcome, a, in terms of increasing subjective value, and the vertical axis represents an ordering of the second outcome, x, in terms of increasing subjective vlaue. Gambles of the form (a, p, r) lie on the horizontal axis, and gambles of the form (r, p, x) lie on the vertical axis; the axes cross at the point (r, r). A p-gamble (a, p, x) is *regular* if $a >_p r >_p x$, or $x >_p r >_p a$. The regular p-gambles are located in the upper left and lower right quadrants of figure 4–2. If (a, p, x) is regular, the value of (a, p, x) is given by the equation,

$$V(a, p, x) = \pi(p)v(a) + \pi(1 - p)v(x). \tag{3}$$

A gamble (a, p, x) is said to be *irregular* if $a \geqslant_p r$ and $x \geqslant_p r$, or $a \leqslant_p r$ and $x \leqslant_p r$. The irregular p-gambles are located in the upper right and lower left quadrants of figure 4–2. To specify the value of an irregular gamble, there are two cases to consider. If $a \geqslant_p x \geqslant_p r$, or $a \leqslant_p x \leqslant_p r$, then the value of (a, p, x) is given by the equation,

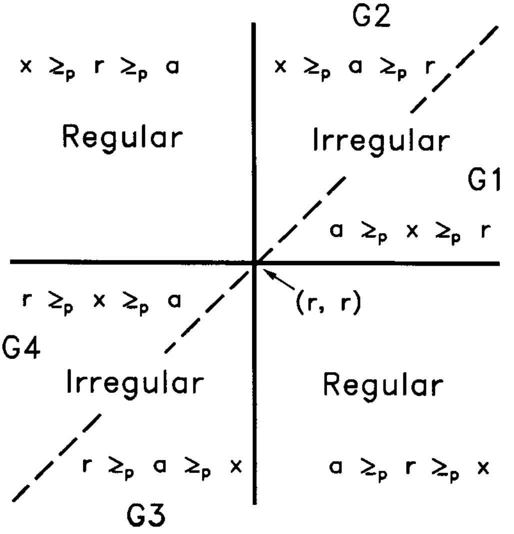

Figure 4–2. A gamble of the form, (a, p, x), is assigned the coordinates, (a, x). Quadrants are labeled according to whether the gambles are regular or irregular. G1, G2, G3, and G4 denote subsets of irregular gambles (see text).

$$V(a, p, x) = \pi(p)v(a) + [1 - \pi(p)]v(x). \tag{4}$$

This rule, equation (4), applies to gambles in subquadrants G1 and G4 of figure 4–2. If $x \geqslant_p a \geqslant_p r$ or $x \leqslant_p a \leqslant_p r$, then the value of (a, p, x) is given by the equation,[2]

$$V(a, p, x) = [1 - \pi(1 - p)]v(a) + \pi(1 - p)v(x). \tag{5}$$

This rule, equation (5), applies to gambles in subquadrants G2 and G3 of figure 4–2. It is not possible to review here the arguments for applying different rules in the evaluation of these gambles (Kahneman and Tversky, 1979, 1984; Miyamoto, 1987). For present purposes, what is important is that prospect theory postulates one rule for regular prospects, a second rule for the irregular gambles in G1 ∪ G4, and a third rule for the irregular gambles in G2 ∪ G3.

I should point out that the definition of regular and irregular p-gambles adopted here is slightly different from Kahneman and Tversky's (1979) definition. If a or x equals r, then Kahneman and Tversky classify (a, p, x) as a regular gamble and use equation (3) to determine $V(a, p, x)$, but the present definitions classify (a, p, x) as an irregular gamble and use equations (4) or (5) to calculate $V(a, p, x)$. This difference has no effect on substantive theory, because if a or x equal r, the value of $V(a, p, x)$ is the same regardless of whether (a, p, x) is classified as a regular or irregular gamble. In other words, if (a, p, x) lies on the vertical or horizontal axis of figure 4–2, the value of $V(a, p, x)$ remains the same if the value is computed by the rule for either adjacent quadrant. The advantage of altering Kahneman and Tversky's classification of gambles is that it simplifies the discussion of MAUT analyses without altering any of the empirical claims of prospect theory (see note 6).

It should be obvious that prospect theory implies that generic utility theory is satisfied by the gambles *within* the sets G1, G2, G3, and G4, for these are triangular sets of p-gambles, and equations (4) and (5) imply that the preference orderings within these sets satisfy the generic utility representation, condition (2), with possibly different coefficients for different subsets. Prospect theory also implies that the standard model for p-gambles is violated, at least for some values of p. The full argument for this cannot be given here, but it rests on the fact that the prospect theoretic analysis of the Allais paradox implies that $\pi(p) \neq 1 - \pi(1 - p)$ for at least some p. For any such p, the standard model for p-gambles is violated, because there is no single pair of coefficients that satisfies the standard model with respect to every gamble in $G1 \cup G2$. To see this, note that if (a, p, x) is in G1, then $V(a, p, x)$ is given by equation (4), whereas if (a, p, x) is in G2, then $V(a, p, x)$ is given by equation (5). The coefficient for $v(a)$ is either $\pi(p)$ or $1 - \pi(1 - p)$, depending on whether (a, p, x) is in G1 or G2, thus violating the standard model for p-gambles which requires that these coefficients be identical for a given choice of p. A similar argument shows that there is no single pair of coefficients that satisfies the standard model with respect to every gamble in $G3 \cup G4$. Thus, prospect theory implies that the generic utility representation is satisfied by the preference ordering within subsets of the form G1, G2, G3 or G4, but it denies that the standard model for p-gambles is satisfied by the preference order over $G1 \cup G2$, or $G3 \cup G4$, at least for some values of p. The example of prospect theory shows why it is desirable to generalize the standard model for p-gambles to the generic utility representation.

Before leaving the discussion of prospect theory, I should mention a final technical point. Prospect theory implies that the preference orderings on G1, G2, G3, and G4 satisfy the generic utility representation, condition (2), but it does not imply that the preference orderings on these subsets satisfy all of the axioms for generic utility theory. The reason is that some of the axioms are not logically necessary, that is, they are not implied by the generic utility representation. The three nonnecessary axioms for the generic utility representation are all existential axioms, that is, axioms that assert the existence of gambles of specified forms (see Axioms 8 to 10 of Appendix 1). Axiom 8 is a restricted solvability assumption similar to solvability assumptions in additive conjoint measurement (Krantz et al., 1971, chapter 6). Axiom 9 makes the innocuous claim that there exist x and y such that $x >_p y$, and Axiom 10 asserts that if $x >_p y$ then there exists z such that $x >_p z >_p y$. The existential assumptions are technical assumptions that are usually not tested empirically. They are usually accepted or rejected on theoretical grounds, depending on whether the topological conditions that they formalize are reasonable in the intended interpretation. The main empirical axioms of generic utility theory are necessary assumptions in the sense that they are implied by condition (2). As prospect theory implies condition (2), it implies the main empirical axioms of generic utility theory, but it does not imply the existential axioms of the theory.[3]

Compatibility of Utility Theories with Generic Utility Theory

The example of prospect theory illustrates what is at issue when asking whether a utility theory is compatible with generic utility theory. A theory is compatible with generic utility theory if it implies that condition (2) is satisfied with respect to a triangular set of p-gambles. If a theory implies (2), it implies the main empirical axioms of generic utility theory. Although it need not imply the existential axioms of generic utility theory, these axioms are very plausible whenever the outcome domain includes attributes like money or survival duration that vary continuously in value. Table 4–1 lists theories that are compatible or incompatible with generic utility theory in this sense. All of these theories describe preferences for more general classes of gambles than the p-gambles mentioned in table 4–1, but only the representation of p-gambles is relevant to the present discussion.

Obviously, any utility theory that implies the standard model for p-gambles, condition (1), is compatible with generic utility theory. Therefore EU and SEU theory and Karmarkar's (1978) SWU theory are combatible with generic utility theory. Edwards' (1962) ASEU and NASEU models postulate that a single pair of coefficients weight the

Table 4–1. Compatability Between Other Theories and Generic Utility Theory.

Compatible Theories	*Functional Form for U(a, p, x), where* $1 > p > 0$, *and* $a \leqslant_p x$.	*Comments*
Expected utility (EU)	$pU(a) + (1 - p)U(x)$	1, 2
Additive subjective expected utility (ASEU)	$w(p)U(a) + [c - w(p)]U(x)$	1, 3
Non-additive subjective expected utility (NASEU)	$w_1(p)U(a) + w_2(1 - p)U(x)$	1, 4
Subjectively weighted utility model (SWU)	$w(p)]U(a) + [1 - w(p)]U(x)$	1, 5
Prospect theory	$\pi(p)U(a) + [1 - \pi(p)]U(x)$ if $x \leqslant_p r$, $[1 - \pi(1 - p)]U(a) + \pi(1 - p)U(x)$ if $r \leqslant_p a$.	6, 7
Dual bilinear model	$S^-(p)U(a) + [1 - S^-(p)]U(x)$	6, 8
Rank dependent utility	$g(p)U(a) + [1 - g(p)]U(x)$	6, 9
Incompatible Theories		
Weighted utility	$\dfrac{pW(a)U(a) + (1 - p)W(x)U(x)}{pW(a) + (1 - p)W(x)}$	1, 10
Lottery dependent expected utility	$pU(a, c_F) + (1 - p)U(x, c_F)$	1, 11

Comments:
1. The functional form is identical when $a \geqslant_p x$.
2. Luce and Raiffa (1957), von Neumann and Morgenstern (1944).
3. c is a positive constant, and $c > w(p) > 0$ (Edwards, 1962).
4. $w_1(p) > 0$, $w_2(p) > 0$ (Edwards, 1962).
5. $w(p) = p^a/[p^a + (1 - p)^a]$, for some a (Karmarkar, 1978).
6. The functional form for $U(a, p, x)$ is analogous, but not identical, in the case where where $a \geqslant_p x$.
7. $1 > \pi(p) > 0$ if $1 > p > 0$ (Kahneman and Tversky, 1979).
8. $1 > S^-(p) > 0$ if $1 > p > 0$ (Luce and Narens 1985; Luce, 1990; Narens and Luce, 1986).
9. $1 > g(p) > 0$ if $1 > p > 0$ (Chew, 1984; Luce, 1988a; Segal, 1984).
10. $W(a)$, $W(x) > 0$ (Chew and MacCrimmon, 1979; Chew, 1983; Fishburn, 1983).
11. $c_F = H(a, p, x)$, for some real-valued function H (Becker and Sarin, 1987; Sarin, to appear).

utilities of outcomes over the full set of p-gambles, but they do not require that the coefficients sum to one. Because the generic utility representation does not require that the coefficients sum to one, these models are also compatible with generic utility theory. As noted above, prospect theory implies that the generic utility representation (2) is satisfied by any triangular set of p-gambles for nongains or for nonlosses. The dual bilinear model (Luce and Narens, 1985; Luce, 1990) and rank dependent utility theory (Chew, 1984; Luce, 1988a; Quiggin, 1982; Segal, 1984; Yaari, 1987) imply that the generic utility representation is satisfied on any triangular set of p-gambles. Unlike prospect theory, these theories do not postulate that the utility representations differ depending on whether gamble outcomes are more or less preferred than a reference level. Recently, Luce (1990, 1992) has formulated a rank and sign dependent utility theory that integrates the distinction between gains and losses into the structure of rank dependent utility theory. Although this theory cannot be discussed here, its relation to generic utility theory is much like that of prospect theory. Rank and sign dependent utility theory implies the generic utility representation (2) with respect to any triangular set of p-gambles with nongain or nonloss outcomes.

Weighted utility theory (Chew and MacCrimmon, 1979; Chew, 1983; Fishburn, 1983) and lottery dependent utility theory (Becker and Sarin, 1987; Sarin, 1992) are two theories that do not imply the generic utility representation (2) except in special cases where these theories reduce to EU theory. I will not demonstrate the incompatibility of weighted utility theory and lottery dependent utility theory with generic utility theory, but the incompatibility is proved by finding utility and weighting functions that satisfy the axioms of these theories, while nevertheless implying violations of the axioms for generic utility theory. Intuitively, the reason these theories are incompatible with generic utility theory is that these theories do not satisfy what Machina (1989) calls replacement separability. A utility theory satisfies replacement separability (with respect to two-outcome gambles) if there exist functions F_1 and F_2 such that $U(a, p, x) = F_1(a, p) + F_2(x, 1 - p)$. Generic utility theory requires replacement separability within a triangular set of p-gambles, whereas weighted utility and lottery dependent utility theories do not.

In summary, a utility theory is compatible with generic utility theory if it implies condition (2) with respect to a triangular set of p-gambles. Empirical or theoretical analyses that are conducted within generic utility theory will be interpretable from the standpoint of any theory that is compatible with generic utility theory. This does not exclude the possibility that analyses in the generic utility framework will be informative from the standpoint of theories that are incompatible with it, but the interpretation from these standpoints will be less straightforward.

Empirical Application of Generic Utility Theory

Next, a generic utility analysis of a MAUT model will be presented to show concretely why such analyses are interpretable from diverse theoretical standpoints. The topic of this analysis, the utility of survival duration and health quality, is of importance and independent interest in the decision analysis of medical therapy selection (Loomes and McKenzie, 1989; McNeil and Pauker, 1982; Miyamoto and Eraker, 1985; Weinstein et al., 1980). Suppose that (Y, Q) denotes a health outcome consisting of Y years of survival in health state Q, followed by death at the end of the Y-th year. The health state, Q, is assumed to be approximately constant during the Y years of survival. The problem investigated here is that of determining the form of the joint utility function $U(Y, Q)$. This is a special case of the problem of determining the utility of time streams of health states. A time stream of health states is a sequence, $(Q_1, Q_2, Q_3. \ldots)$, where each Q_i represents the health quality during the i-th time period. The problem of time streams will not be discussed here (compare Stevenson, 1986; Pliskin, Shepard, and Weinstein, 1980; Mehrez and Gafni, 1989).

Pliskin et al. (1980) formalized various health utility models, including multiplicative and additive models for the combination of duration and quality.[4] The utility of Y and Q is multiplicative if there exist utility scales for duration and quality, denoted F and G, respectively, such that

$$U(Y, Q) = F(Y) \cdot G(Q), \tag{6}$$

for every Y and Q. The utility of Y and Q is additive if there exist utility scales for duration and quality, denoted F' and G', respectively, such that

$$U(Y, Q) = F'(Y) + G'(Q), \tag{7}$$

for every Y and Q. It is well known that if EU theory is satisfied and the attributes Y and Q are mutually utility independent, then the bivariate utility function $U(Y, Q)$ must be multiplicative or additive (Keeney and Raiffa, 1976). Assuming EU theory, the additive model can be distinguished from the multiplicative model by the fact that the additive model implies the marginality property: It implies that gambles with identical marginal probability distributions over attributes are equal in preference (Fishburn, 1965). Thus, gambles A and B should be equally preferred if the utility model is additive and EU theory is valid.

Gamble A	*Gamble B*
50% chance, 25 years, pain	50% chance, 25 years, no pain
50% chance, 3 years, no pain	50% chance, 3 years, pain

Pliskin et al. pointed out that Gamble B is usually preferred to Gamble A and concluded that the additive utility model must be rejected. The multiplicative model is consistent with violations of marginality. Therefore, Pliskin et al. proposed that the utility of duration and quality is described by a multiplicative model, basing their hypothesis on the assumptions that EU theory is valid, that duration and quality are mutually utility independent, and that marginality is violated. They did not test empirically whether mutual utility independence was satisfied.

Working within the generic utility framework, Miyamoto and Eraker (1988) proposed an alternative formalization of the multiplicative model. They noted that survival duration and quality appear to be sign dependent attributes in the sense of Krantz et al. (1971). The concept of sign dependence is illustrated by the following examples. One generally prefers longer survival to shorter survival, but if the health state is exceptionally bad, one prefers shorter survival to longer survival. Exceptionally bad quality inverts the normal preference order over survival duration as if $G(Q) < 0$ for some Q, and $U(Y, Q) = F(Y) \cdot G(Q)$. Another significant fact is that one normally prefers good health to poor health, for example, 2 years in good health is preferred to 2 years in poor health, but one has no preference between 0 years in good health and 0 years in poor health. Thus, immediate death nullifies the preference order over health quality, as if $F(0) = 0$, and $U(0, Q) = F(0) \cdot G(Q) = 0$ for any choice of Q. The sign dependence of survival duration and health quality is diagnostic of a multiplicative utility model, but it is not sufficient for it. Miyamoto (1985) and Miyamoto and Eraker (1988) formulated an axiomatization of the multiplicative health utility model within the generic utility framework. The discussion of the axiomatization will be more straightforward if we only consider health states that are better than death. The utility analysis of worse-than-death health states involves complications that are irrelevant to our present purpose, which is to exemplify the generic approach to utility modeling. In presenting a generic utility formalization, we must choose whether to state axioms in terms of an upper or lower triangular set of p-gambles. As either choice is equally useful, I will arbitrarily choose to develop the axiomatization in terms of an upper triangular set.

Assuming, then, that the health states under investigation are all better than death and that the axioms for the generic utility representation are valid, one first postulates that survival duration is utility independent from health quality in the sense that preferences among gambles for survival duration are the same for any fixed choice of health quality. The following definition states the utility independence property within the generic utility framework:

DEFINITION 1. Suppose that the set of consequences, C, is the Cartesian product of a set of survival durations, D, and a set of health states, S, in other words, $C = D \times S$. Then, *survival duration is utility independent from health quality* iff the following equivalence holds: For every Y_1, Y_2, Y_3, $Y_4 \in D$ and Q_1, $Q_2 \in S$,

$$[(Y_1, Q_1), p, (Y_2, Q_1)] \geqslant_p [(Y_3, Q_1), p, (Y_4, Q_1)]$$
$$\text{iff}$$
$$[(Y_1, Q_2), p, (Y_2, Q_2)] \geqslant_p [(Y_3, Q_2), p, (Y_4, Q_2)]. \quad (8)$$

whenever $(Y_2, Q_1) \geqslant_p (Y_1, Q_1)$, $(Y_4, Q_1) \geqslant_p (Y_3, Q_1)$, $(Y_2, Q_2) \geqslant_p (Y_1, Q_2)$, and $(Y_4, Q_2) \geqslant_p (Y_3, Q_2)$.

Definition 1 differs from the standard EU formulation of utility independence (Keeney and Raiffa, 1976) only insofar as it stipulates that all of the gambles in the independence relation must be members of an upper triangular set. Definition 1 is strictly weaker than the standard EU formulation of utility independence because the latter claims that the property is satisfied by all p-gambles, or by all gambles generally, depending on the formulation. The utility independence of health quality from survival duration is defined analogously to Definition 1 with the role of survival duration and health quality interchanged.

Just as in EU theory, generic utility theory implies that if survival duration and health quality are each utility independent from the other, then the joint utility function $U(Y, Q)$ is either multiplicative or additive (Miyamoto, 1988). In the present case, however, we do not need to postulate the utility independence of health quality from survival duration because the sign dependence relations between duration and quality are sufficient (in combination with the utility independence of duration from quality) to establish the validity of the multiplicative model. In particular, I adopt the assumption that immediate death nullifies the preference ordering over health quality; stated formally,

$$(0, Q_1) \sim_p (0, Q_2) \tag{9}$$

for every Q_1, $Q_2 \in S$. Miyamoto (1985) proved the following theorem:

THEOREM 1. Suppose that the set of consequences, C, is the Cartesian product of a set of survival durations, D, and a set of health states, S; suppose that $R(p)$ is a triangular set of p-gambles with outcomes in C; suppose that $\geqslant_p$ is a relation on $R(p)$; and suppose that the structure $(C, p, R(p), \geqslant_p)$ satisfies the axioms for the generic utility representation (Appendix 1). If survival duration is utility independent from health quality (Definition 1), and if immediate death nullifies the preference order over health quality (equation (9)), then there exist scales U, F and G such that U preserves the preference order in the sense of equation (2), and

$$U(Y, Q) = F(Y) \cdot G(Q)$$

for every $Y \in D$ and $Q \in S$.

Theorem 1 is proved in Appendix 2. An interesting technical point regarding the assumptions of Theorem 1 is that the set of health states is allowed to be finite. The theorem applies even if there are only two health states in the set S. The assumptions of generic utility theory require that the set of consequences $C = D \times S$ be infinite, but this is satisfied because the set of possible survival durations is infinite.[5] From a practical standpoint, it is easier to test an axiomatization if the test does not require large numbers of different health qualities because it is time consuming to explain a large variety of health qualities to subjects. The present axiomatization allows us to restrict attention to a small set, S, of health states. Of course, even if the aximoatization is empirically supported with respect to the health states in S, it may be violated by preferences for health states that are not in S. In this case, the multiplicative model would have valid for the health states in S, but not for states outside of S.

Miyamoto and Eraker (1988) assumed that condition (9) was introspectively obvious and undertook to test the utility independence of survival duration from health quality in a sample of medical patients. Subjects were inpatients at the Ann Arbor VA Medical Center and the University of Michigan Hospital. Subjects included patients with cancer, heart disease, diabetes, arthritis, and other serious ailments. Each subject was asked to compare two health states, referred to as *survival with current symptoms* and *survival free from current symptoms*. "Current symptoms" was defined to be health symptoms at the severity and frequency experienced by the subject during the month preceding the interview. "Freedom from current symptoms" was simply survival without the health problems that comprised current symptoms. To be a subject in the experiment, a patient had to satisfy two criteria. First, it was required that subjects preferred twenty-four years of survival without current symptoms to twenty-five years of survival with current symptoms. Subjects who preferred twenty-five years of survival with current symptoms to twenty-four years without the symptoms were not included in the sample. Willingness to give up at least one year out of twenty-five in order to be free from their symptoms constituted an operational criterion for the claim that subjects regarded their health symptoms as severe. Second, it was required that every subject always preferred additional survival, up to twenty-five years, even if current symptoms prevailed. Thus, subjects were chosen who satisfied the assumption that the health states under investigation were better than death. Although utility modeling in the domain of health must ultimately analyze the impact of worse-than-death health states, this issue was avoided in the present study.

The main issue in the experiment was whether the certainty equivalents of gambles for survival duration would differ for survival with or without current symptoms. All gambles used as stimuli were even-chance gambles between a shorter and longer duration of survival. Hence, the stimulus gambles were drawn from an upper triangular set. In order to interpret data from the standpoint of prospect theory, each subject was asked to state his or her own reference level for survival duration. The concept of a reference level was explained to the subject as follow:

> I'm going to ask you about something called the aspiration level for survival. Since this concept is fairly complicated, I'll explain it in several steps. The aspiration level for survival is defined to be the length of survival that marks the boundary between those survivals that you regard as a loss and those survivals that you regard as a gain.
>
> For example, my own aspiration level for survival is about the age of sixty. This means that if I found out that I were going to live to the age of fifty or fifty-five (but no more), I would regard this as something of a loss. If I found out that I were going to live to sixty-five or seventy, I would regard this as something of a gain. The aspiration level for survival is not the same as my life expectancy, since my life expectancy is greater than sixty. It's also not the length of time I would want to live, since if I were in good health, I would want to live at least to eighty. The age of sixty is simply a target that marks the boundary between survivals that I would regard to some degree as a loss and survivals that I would regard to some degree as a gain.

> I should mention that there's nothing special about the age of sixty. Some individuals place their aspiration level at a very large number, like ninety. For such a person, any survival less than the age of ninety would be regarded to some degree as a loss. I've also encountered individuals who set their aspiration level for survival at their present age. This does not mean that they no longer want to live. It means that they regard every year of survival as a gain. If such an individual learned that he had two years to live, he would regard this as gaining two years of survival, rather than to emphasize some longer survival of which he's being deprived.
>
> Does this concept of an aspiration level of survival make sense to you? Can you tell me what your own aspiration level for survival is?

Subjects generally found these instructions meaningful, and would state a reference level without appearing to be confused.

The stimulus gambles in the experiment were even-chance gambles (p-gambles with $p = 0.5$) for which the second outcome was always greater than the first. Thus, the stimulus gambles were drawn from an upper triangular set of 0.5-gambles. The outcomes in the stimulus gambles ranged from zero years (immediate death) to a maximum of twenty-four years. A complete description of the stimulus gambles is given in Miyamoto and Eraker (1988). Each subject judged the certainty equivalents of six gambles for survival duration. The judgments were elicited in a block under the assumption that survival was accompanied by current symptoms, and in a second block under the assumption that survival was free from current symptoms. The relative order of the two blocks was counterbalanced across subjects. The two blocks of judgments were replicated on a second day with the health qualities associated with the blocks in the same order on the second day as on the first day. Thus, the experimental design within each subject was a 6×2 ANOVA in which the factors were gamble (6 levels) and health state (2 levels); there were two replications per cell in the ANOVA.

Earlier, I pointed out that prospect theory implies that the generic utility representation is satisfied by a triangular set of gambles, if every outcome is at least as preferred as the reference level, or if every outcome is equal or less preferred than the reference level. Assuming that self-reported reference levels were valid, we can determine whether a subject's reference level satisfied this requirement relative to the stimulus gambles of the experiment. The shortest duration in any stimulus gamble was zero years, and the longest duration was twenty-four years. For any subject whose reference level was his present age, the stimulus gambles were drawn from an upper triangular set of nonloss gambles, like the set

G2 in figure 4–2. For any subject whose reference level was equal or greater than present age plus twenty-four years, the stimulus gambles were drawn from an upper triangular set of nongain gambles, like the set G4 of figure 4–2. Subjects who fell into these two classes will be called *purely irregular subjects* because every stimulus gamble was irregular relative to their reference levels.[6] Prospect theory predicts that the preferences of purely irregular subjects satisfy generic utility theory. Hence, if the utility independence of survival duration from health quality is tested in the preferences of purely irregular subjects, the results of the test are interpretable from the standpoint of prospect theory, as well as from other theoretical standpoints.

There were twenty-seven subjects, seventeen of whom were purely irregular. From the standpoint of prospect theory, the analysis of the response of subjects who were not purely irregular is extremely complicated. For these subjects different, stimulus gambles were regular or irregular, depending on the value of the subject's reference level, and different analyses would be required for the regular and irregular gambles; furthermore, the division of gambles into regular and irregular gambles differed from subject to subject because subjects differed in their reference levels. Because of these complications, results will be presented only for the seventeen purely irregular subjects. A more comprehensive analysis of the data for all twenty-seven subjects is presented in Miyamoto and Eraker (1988). Among the seventeen purely irregular subjects, three subjects set the reference level at their present ages; these subjects will be referred to as low reference level subjects. Fourteen subjects set the reference level at a point equal or beyond present age plus twenty-four years; these subjects will be referred to as high reference level subjects. Descriptive statistics are presented in Table 2 for low and high reference level subjects, and for the two groups combined.

For simplicity, the condition where survivals were assumed to be accompanied by current symptoms will be called the "poor health" condition, and the condition where survivals were assumed to be free from current symptoms will be called the "good health" condition. The expressions "good health" and "poor health" were not used to designate these conditions when discussing them with subjects. The dependent measure, the certainty equivalents of gambles, was transformed prior to statistical analysis. To define the transformation, let CE denote the judged certainty equivalent of a gamble between a shorter duration (LOW) and a longer duration (HIGH). The assumed health state could be either poor health or good health. The transformed response, denoted PE, was computed by the rule,

Table 4–2. Descriptive Statistics for the Purely Irregular Subjects.

	Low Ref. Level n = 3		*High Ref. Level n = 14*		*All Subjects n = 17*	
	Mean	SD	Mean	SD	Mean	SD
Age	29.7	5.0	32.4	6.1	31.9	5.9
Ref. Level	29.7	5.0	66.9	6.7	60.3	15.9
Proportional Equivalents						
Good Health	0.53	0.02	0.55	0.20	0.54	0.18
Poor Health	0.63	0.18	0.56	0.21	0.57	0.21

$$PE = \frac{CE - LOW}{HIGH - LOW}. \tag{10}$$

Note that HIGH—LOW represents the range of the stimulus gamble. Therefore the transformed response represents the proportion of the range of the gamble that was exceeded by the certainty equivalent. The transformed response will be referred to as a proportional equivalent. There are two main advantages to using proportional equivalents in the analysis. First, the variance of a certainty equivalence judgment generally increases as the range of the stimulus gamble increases. Transformation of certainty equivalents to proportional equivalents tends to equalize the variance within different cells of the ANOVA. Second, mean proportional equivalents are more easily interpreted than mean certainty equivalents. For example, if the mean proportional equivalent were found to be 0.55, one may infer that the average certainty equivalent was slightly greater than the expected value of the gamble, but if the mean certainty equivalent were found to be 12.3 years, the result could not be interpreted without examining the specific durations that were used in the stimulus gambles. Table 4–2 contains the mean proportional equivalents in the good health and poor health conditions. On the average, subjects were close to being risk neutral, with a slight (nonsignificant) tendency to be risk seeking.

The utility independence of survival duration from health quality predicts that mean certainty equivalents in the good health condition should equal mean certainty equivalents in the poor health condition. Furthermore, there should be no interaction between the good health/poor health distinction and the specific gamble being tested because equality in certainty equivalents is predicted to hold for each gamble individually. Since proportional equivalents are linearly related to certainty equivalents, utility independence implies the same predictions for proportional equivalents. In other words, if utility independence is satisfied, the ANOVA performed on proportional equivalents should have no main effect of health quality and no interaction between survival duration and health quality. Of course, some significant effects should be observed even if utility independence is satisfied because false rejections of null hypotheses (Type I errors) are a necessary consequence of random variation in judgments. Nevertheless, such rejections should not occur more frequently than the significance level of the test, nor should there be a qualitative pattern to such rejections.

A two-factor ANOVA was computed within the data of each subject. Five of the seventeen subjects had significant ($p < 0.05$) main effects for health state. If the null hypothesis for the main effect were true in all seventeen tests, the chance of five or more rejections would be less than 0.005 (computed as 1 minus the cumulative binomial probability of 4 or fewer rejections given 17 independent chances for rejecting at the 0.05 level). Therefore the observed number of significant main effects was inconsistent with utility independence. One subject had a significant ($p < 0.01$) interaction between health quality and gamble. If the true chance of rejecting the null hypothesis for interaction were 0.01, the chance of 1 or more rejections would be greater than 0.15. Therefore the observed number of significant rejections of the interaction hypothesis was compatible with utility independence. Figure 4–3 shows a scatter plot of mean proportional equivalents in the good health versus poor health conditions. Plus signs indicate subjects whose mean proportional equivalents were significantly different in the good health and poor health conditions. Asterisks indicate subjects whose mean proportional equivalents were not significantly different. The scatter plot indicates that the mean proportional equivalents of most subjects were close to equality in the two conditions. Even among subjects who significantly violated utility independence, the change in certainty equivalents as a function of assumed health state was generally smaller than 20 percent of the range in the gambles. Miyamoto and Eraker (1988) carried out an extensive power analysis of the test of utility independence. They found that the tests of the main effect of health quality were sufficiently strong to detect true effects that were greater than ± 0.10, but true effects smaller than ± 0.05 would have been difficult to detect in this experiment.

We may conclude from this analysis that at least some subjects violated utility independence of survival duration from health quality, but the departures were generally small; the majority of subjects were close

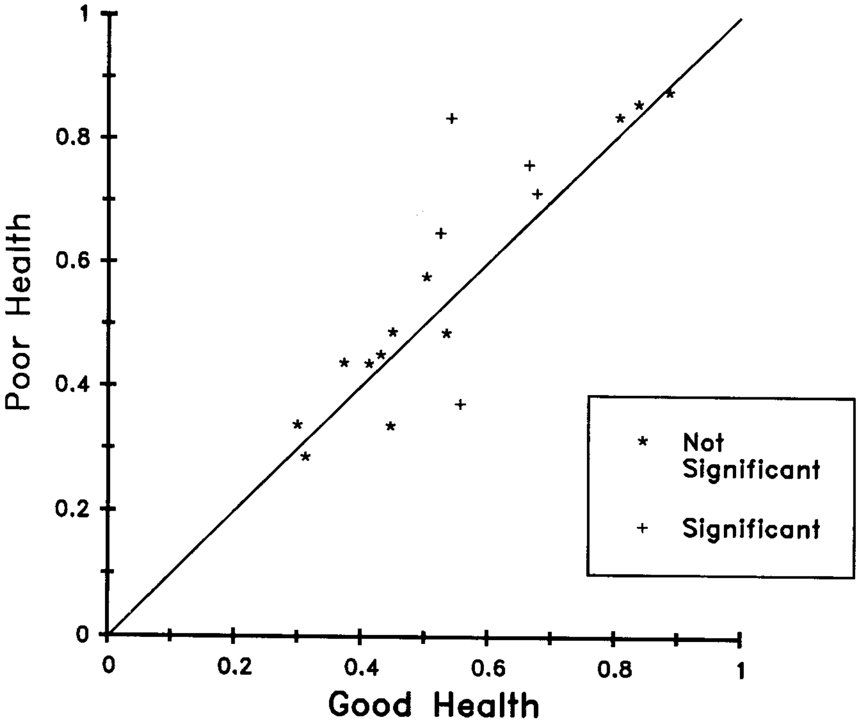

Figure 4–3. Scatter plot of mean proportional equivalents in the good health and poor health conditions.

to the predicted equality between good and poor health conditions. Assuming that generic utility theory is valid, and that immediate death nullifies the preference order over health quality, we may conclude that the preferences of most subjects satisfy a multiplicative utility model with respect to survival with and without current symptoms. Furthermore, the multiplicative utility model appears to be a good approximation even where it was violated.

The key issue in this discussion is not the validity of the multiplicative utility model per se, rather the purpose of this exercise is to illustrate the methodology of utility modeling within the generic utility framework. The stimulus gambles were selected from a triangular set of even-chance gambles. In addition, a subset of subjects was identified for whom every stimulus gamble was an irregular gamble relative to their reference levels.

Given these design characteristics, prospect theory predicts that the generic utility representation must be satisfied with respect to the stimulus gambles of the experiment. Therefore, the preceding interpretation of experimental results remains the same if generic utility theory is replaced by prospect theory as the theoretical framework of the analysis. The interpretation also remains the same in the framework of other theories that imply generic utility theory, for example, from the standpoint of EU and SEU theory, the dual bilinear model, rank-dependent utility theory, Karmarkar's SWU theory, and Edwards' ASEU and NASEU theories. Because these theories are compatible with generic utility theory, the preference assumptions stated in Theorem 1 continue to imply the multiplicative utility model, if generic utility theory is replaced by one of these stronger theories, and hence, the experimental test of utility independence has the same implications for health utility structure in the framework of these stronger theories. In this sense, the experimental analysis presented here is a generic analysis of a utility model, for its interpretation remains the same from diverse theoretical perspectives.

MAUT in a Generic Framework

It is not hard to show that additive, multiplicative, and multilinear utility models of arbitrarily many attributes can be axiomatized within the generic utility framework. Miyamoto (1988) axiomatized the additive and multiplicative utility models in the two-attribute case, and the generalizations to arbitrarily many attributes is straightforward. Log/power utility functions and linear/exponential utility functions can also be axiomatized within the generic utility framework, and experimental tests of these axiomatizations are also straightforward (Miyamoto, 1988; Miyamoto and Eraker, 1989). For the sake of brevity, these axiomatizations will not be stated here, but they follow the pattern displayed in Definition 1—standard MAUT assumptions are restricted to the preference order over a triangular set of *p*-gambles, and the implications of these assumptions are found to be the same in the generic utility framework as in the EU framework. The only exception to this claim is the axiomatization of the additive utility model. The marginality assumption that is used to axiomatize the additive model in the EU framework cannot be translated into the generic utility framework because its implications are derived under the assumption that the utilities of outcomes are weighted by the stated probabilities in a gamble, and generic utility theory allows this assumption to be violated. Miyamoto (1988) discovered a simple, testable axiomatization of the additive model in the generic utility framework, but this axiomatization will not be presented here.

The methodology for testing axiomatizations of utility models in the generic utility framework is generally like the experimental example presented here (Miyamoto, 1988; Miyamoto and Eraker, 1988, 1989). One formulates the appropriate independence assumptions in terms of a triangular set of p-gambles, and tests of these assumptions can be interpreted from the standpoint of any theory that implies the generic utility representation. If one intends to interpret results from the standpoint of prospect theory, one must employ some procedure for identifying reference levels. The results for purely irregular subjects are interpretable within prospect theory as well as within other theories that imply generic utility theory.

Although *testing* utility models is no more difficult in the generic utility framework than in the framework of EU theory, it should be pointed out that *scaling* utility functions for the given outcome domain is actually more complicated in the generic utility framework. Whereas EU theory postulates that $U(a, p, x) = pU(a) + (1 - p)U(x)$, where p and $1 - p$ are known because they are simply the stated probabilities of the outcomes, generic utility theory postulates that $U(a, p, x) = \alpha U(a) + \beta U(x)$, where α and β are unknown positive weights. Empirical scaling of utility functions is simpler in the EU framework because p and $1 - p$ are assumed to weight the utility of outcomes. Scaling utility functions in the generic utility framework will require the estimation of α and β for each individual.

Conclusions

At the outset of this chapter, it was stated that a major goal was to show that risk-based methods of MAUT modeling are not seriously undermined by the discovery of strong evidence against the descriptive validity of EU and SEU theory. It should now be clear why this is the case. Risk-based methods of MAUT modeling can be logically justified on much weaker assumptions than those of EU and SEU theory, and in particular, they can be justified under the assumptions of generic utility theory. In other words, axioms that imply specific utility models under EU or SEU assumptions imply these same classes of models under the assumptions of generic utility theory (Miyamoto, 1988). The only exception to this claim is Fishburn's (1965) characterization of the additive utility model in terms of marginal probability distributions, but the additive utility model has an alternative, easily tested axiomatization within the generic utility framework (Miyamoto, 1988). Therefore risk-based MAUT methods, which use axioms as diagnostic criteria for utility models in specific domains, can continue to be applied to descriptive modeling with only minor modifications in the implementation of these methods.

Generic utility theory is a useful framework for utility modeling because it is compatible with important non-EU theories that have been proposed as revisions or replacements of EU and SEU theory. Formalizations and empirical tests of utility models within generic utility theory are interpretable from the standpoint of stronger theories, which may not even be consistent with each other. Clearly, this is an advantage when one's primary interest is to determine the structure of utility within a specific domain, rather than to discover which fundamental theory is valid. As I have tried to stress, generic utility theory is a framework for utility modeling; it is not intended as a general foundation for preference under risk. Indeed, the very limitations that make it a useful tool for utility modeling also render it utterly inadequate as a general foundation. It is hoped that generic utility theory will stimulate useful investigations of empirical utility structures even while fundamental theoretical issues continue to be debated.

The formalization of MAUT models within the generic utility framework demonstrates that risk-based approaches to utility modeling can be imported into prospect theory, the dual bilinear model, and rank dependent utility theories (Miyamoto, 1988). This result is of independent importance, for it has not previously been shown how to develop MAUT axiomatizations in these theoretical frameworks. Especially in view of the current interest in non-EU theory development, it is reassuring to know that risk-based MAUT methods can be incorporated into such theories without extensive revision. We see here a third advantage of generic utility theory—any axiomatization that is developed within the generic utility framework automatically transfers to stronger theories that imply it. Thus, if one wants to determine whether a descriptive theory allows the risk-based formalization of standard MAUT models, one can check whether it implies the generic utility representation on a triangular set of p-gambles. An affirmative determination establishes that risk-based MAUT formalizations can be developed within the theory. A negative determination does not exclude the possibility that risk-based MAUT formalizations can be developed within the theory, but the methodology described here will not apply.

Acknowledgments

I would like to thank Ward Edwards, Earl Hunt, Irving LaValle, and Leigh Thompson for helpful comments on an earlier draft of this chapter. Conversations with Colin Camerer and Peter Wakker improved my understanding of current work in utility theory. Of course, I am solely responsible for inadequacies of the present investigations.

Appendix 1: Axiomatization of Generic Utility Theory and the Representation and Uniqueness Theorem

Let C be a nonempty set of consequences; let $R(p)$ denote a set of p-gambles with outcomes in C. Let $\geqslant_p$ denote the preference relation, defined on outcomes in C and also on gambles in $R(p)$. The axiomatization of generic utility theory will be stated in terms of an upper triangular set. The axiomatization presented here is similar to an axiomatization developed in Miyamoto (1988), but the present treatment is more explicit and transparent. Both the present axiomatization and Miyamoto (1988) are heavily influenced by Krantz et al. (1971, Chapter 6). Wakker (1989) independently proved a utility representation theorem that includes the generic utility representation as a special case.

Axiom 1 asserts that $\geqslant_p$ is a connected, transitive ordering of C. Axiom 2 asserts that $\geqslant_p$ is also a connected, transitive ordering of $R(p)$. (Remember that I am not distinguishing notationally between the preference order on C and the preference order on $R(p)$). Axiom 3 asserts that $R(p)$ is an upper triangular set.

Axiom 1. For every $a, x \in C$, $a \geqslant_p x$ or $x \geqslant_p a$. For every $x, y, z \in C$, if $x \geqslant_p y$ and $y \geqslant_p z$, then $x \geqslant_p z$.

Axiom 2. For every $(a, p, x), (b, p, y) \in R(p)$, either $(a, p, x) \geqslant_p (b, p, y)$ or $(b, p, y) \geqslant_p (a, p, x)$. For every $(a, p, x), (b, p, y), (c, p, z) \in R(p)$, if $(a, p, x) \geqslant_p (b, p, y)$ and $(b, p, y) \geqslant_p (c, p, z)$, then $(a, p, x) \geqslant_p (c, p, z)$.

Axiom 3. For every $a, x \in C$, $(a, p, x) \in R(p)$ iff $x \geqslant_p a$.

The next axiom asserts that preferences for p-gambles satisfy first-order stochastic dominance with respect to the riskless preference order over C.

Axiom 4. For any $a, b, c, d \in C$,

$$b \geqslant_p c \geqslant_p a \quad \text{iff} \quad (a, p, b) \geqslant_p (a, p, c), \tag{11}$$

and

$$d \geqslant_p b \geqslant_p c \quad \text{iff} \quad (b, p, d) \geqslant_p (c, p, d). \tag{12}$$

Axiom 4 implies that the orderings over the first and second components are mutually independent, and monotonically increasing with respect to each other. It should be noted that Axiom 4 is a strictly weaker condition than the independence axiom of expected utility theory which asserts that conditions (11) and (12) hold when a, b, c, and d can be either *lotteries* or consequences (Machina, 1989).

The next axiom is the Thomsen condition of additive conjoint measurement.

AXIOM 5. Let $a, b, c, x, y, z \in C$ be any elements satisfying $a \leqslant_p x$, $b \leqslant_p y$, $b \leqslant_p z$, $c \leqslant_p x$, $a \leqslant_p z$, and $c \leqslant_p y$. If $(a, p, x) \sim_p (b, p, y)$, and $(b, p, z) \sim_p (c, p, x)$, then $(a, p, z) \sim_p (c, p, y)$.

Next I will formulate an Archimedean axiom (Krantz et al., 1971).

DEFINITION 2. Let N be any set of consecutive integers (positive or negative, finite or infinite). A set $\{a_i \in C : i \in \mathbf{N}\}$ is said to be *a standard sequence on the first component* if there exist $x, y \in C$ such that $x \not\sim_p y$, for every $i \in \mathbf{N}$, we have $a_i \leqslant_p x$ and $a_i \leqslant_p y$, and for every $i, i + 1 \in \mathbf{N}$, $(a_i, p, x) \sim_p (a_{i+1}, p, y)$. A standard sequence on the first component is said to be bounded if there exist $u, v, z \in C$ such that $u \leqslant_p z$, $v \leqslant_p z$, $a_i \leqslant_p z$, and $(u, p, z) \geqslant_p (a_i, p, z) \geqslant_p (v, p, z)$ for every $i \in \mathbf{N}$. The definitions of *a standard sequence on the second component* and *a bounded standard sequence on the second component* are perfectly analogous. A standard sequence is said to be finite if N contains finitely many integers.

AXIOM 6. Every bounded standard sequence on their component is finite.

The next axiom is a qualitative condition that guarantees that the utility scales for the first and second components are linear with respect to each other. An analogous assumption is required in the axiomatization of the standard model for p-gambles (see Krantz et al., 1971, Chapter 6, Theorem 15).

AXIOM 7. Suppose that $a, b, c, x, y \in C$ and $a \leqslant_p w$, $b \leqslant_p x$, $b \leqslant_p w$, $c \leqslant_p x$, $y \leqslant_p a$, $z \leqslant_p b$, $y \leqslant_p b$, $z \leqslant_p c$. If $(a, p, w) \sim_p (b, p, x)$, $(b, p, w) \sim_p (c, p, x)$, and $(y, p, a) \sim_p (z, p, b)$, then $(y, p, b) \sim_p (z, p, c)$. If $(y, p, a) \sim_p (z, p, b)$, $(y, p, b) \sim_p (z, p, c)$, and $(a, p, w) \sim_p (b, p, x)$, then $(b, p, w) \sim_p (c, p, x)$.

Finally, we will require some existential assumptions. Axiom 8 is the restricted solvability assumption of additive conjoint measurement (Krantz et al., 1971, ch. 6) stated in terms of the gambles in $R(p)$. Axiom

9 asserts that the riskless preference order is not trivial. Axiom 10 asserts that there are no empty gaps in the preference ordering over outcomes.

Axiom 8. For any $a, b, c, x, y \in C$, if $a \leqslant_p x$, $b \leqslant_p y$, and $c \leqslant_p x$ and $(a, p, x) \geqslant_p (b, p, y) \geqslant_p (c, p, x)$, then there exists $d \in C$ such that $d \leqslant_p x$ and $(d, p, x) \sim_p (b, p, y)$. For any $a, b, x, y, z \in C$, if $a \leqslant_p x$, $b \leqslant_p y$, and $a \leqslant_p z$ and $(a, p, x) \leqslant_p (b, p, y) \leqslant_p (a, p, z)$, then there exists $w \in C$ such that $a \leqslant_p w$ and $(a, p, w) \sim_p (b, p, y)$.

Axiom 9. There exist $x, y \in C$ such that $x >_p y$.

Axiom 10. If $x, y \in C$ and $x >_p y$, then there exists z such that $x >_p z >_p y$.

The following theorem asserts the existence of the generic utility representation and the interval-scale uniqueness of the utility scale.

Theorem 2. (*Representation and Uniqueness Theorem for the Generic Utility Representation*): Let C be a nonempty set of consequences; let $R(\mathrm{p})$ denote a set of p-gambles with outcomes in C; let $\geqslant_p$ denote a relation on C and $R(p)$. If Axioms 1 to 10 are satisfied, then there exists a function $U: C \to$ Reals, and positive constants, α and β, such that

$$(a, p, x) \geqslant_p (b, p, y) \text{ iff } \alpha U(a) + \beta U(x) \geqslant \alpha U(b) + \beta U(y) \qquad (13)$$

for every (a, p, x), $(b, p, y) \in R(p)$. Moreover if U', α' and β' are any other function and constants that satisfy (13), then there exist $\lambda, \tau, \gamma \in Re$ such that $\lambda, \gamma > 0$, $U' = \lambda U + \tau$, $\alpha' = \gamma\alpha$, and $\beta' = \gamma\beta$. In other words, U is an interval scale.

Proof. The proof consists in showing that Axioms 1 to 10 imply the axioms for a lower triangular additive structure defined in Miyamoto (1988). I will only sketch the proof. The following presentation assumes that the reader is familiar with Definition 7 and Theorem 1 of Miyamoto (1988).

Let juxtaposed symbols denote ordered pairs of elements in $C \times C$, that is, $ax \in C \times C$ is a typical element. Define an ordering, $\geqslant_g$, of $C \times C$ by

$$ax \geqslant_g by \text{ iff } (x, p, a), (y, p, b) \in R(p) \text{ and } (x, p, a) \geqslant_p (y, p, b). \qquad (14)$$

The reason that the ordering of elements is inverted in the correspondence of ax to (x, p, a) is that the present axiomatization is formulated in terms of an upper triangular set of p-gambles, and the axiomatization in Miyamoto (1988) was formulated in terms of a lower triangular set. Obviously this difference is substantively unimportant because either formalization can be transformed into the other by a change of notation.

Axiom 4 and (14) imply that

$$a \geqslant_p x \text{ iff } (a, p, a) \geqslant_p (x, p, x) \text{ iff } aa \geqslant_g xx \tag{15}$$

Define $P \subseteq C \times C$ by $ax \in P$ iff $(x, p, a) \in R(p)$. By Axioms 3 and 4 and condition (15), $ax \in P$ iff $aa \geqslant_g xx$, which is the defining characteristic of the set P in Definition 7 of Miyamoto (1988); therefore, we may identify the set P defined here with the set P defined in Definition 7 of Miyamoto (1988). Let $M1 - M10$ denote the 10 axioms stated in Definition 7 of Miyamoto (1988). I claim that $(C, \geqslant_g)$ satisfies $M1 - M10$.

The following implications are obvious, given that conditions (14) and (15) are satisfied. Axiom $M1$ is implied by Axiom 1. Axiom $M2$ is implied by Axioms 2 and 3. Axiom $M3$ is implied by Axiom 2. Axiom $M4$ is implied by Axiom 4. Axiom $M5$ is implied by Axiom 5. Axiom $M6$ is implied by Axiom 7. Axiom $M7$ is implied by Axiom 6. Axiom $M8$ is implied by Axioms 4, 8 and 9. Axiom $M9$ is implied by Axioms 4 and 9. Axiom $M10$ is implied by Axiom 7; therefore, $(C, \geqslant_g)$ satisfies the axioms of Definition 7 in Miyamoto (1988). By Theorem 1 of Miyamoto (1988) there exists a function $\phi: C \to$ Reals and a constant $\lambda > 0$ such that

$$xa \geqslant_g yb \text{ iff } \phi(x) + \lambda\phi(a) \geqslant \phi(y) + \lambda\phi(b) \tag{16}$$

for every $a, b, x, y \in C$ such that $xa, yb \in P$. Define $U = \phi$ and $\alpha = \beta\lambda$ for some $\beta > 0$. Then, for any $(a, p, x), (b, p, y) \in R(p)$,

$$\begin{aligned} (a, p, x) \geqslant_p (b, p, y) \quad &\text{iff} \quad xa \geqslant_g yb && \text{by (14)} \\ &\text{iff} \quad \phi(x) + \lambda\phi(a) \geqslant \phi(y) + \lambda\phi(b) && \text{by (16)} \\ &\text{iff} \quad \alpha U(a) + \beta U(x) \geqslant \alpha U(b) + \beta U(y). \end{aligned}$$

Therefore the generic utility representation, condition (13), is satisfied. The uniqueness of U, α and β follows from the uniqueness result in Theorem 1 of Miyamoto (1988). **Q.E.D.**

Appendix 2: Proof of Theorem 1

The following proof of Theorem 1 is due to Miyamoto (1985). Conditions (8) and (9) are the hypotheses of Theorem 1. I must show that the multiplicative model, equation (6), is satisfied. Choose an arbitrary $Q_0 \in S$, and define a function $J: D \to$ Reals by $J(Y) = U(Y, Q_0)$. Because duration is utility independent of quality, J is linearly related to utility at any other fixed quality, that is,

$$U(Y, Q) = G(Q) \cdot J(Y) + H(Q) \tag{17}$$

for some real valued functions, G and H. Therefore

$$\begin{aligned} J(0) &= U(0, Q_0) = U(0, Q) && \text{by (9)} \\ &= G(Q) \cdot J(0) + H(Q). && \text{by (17)} \end{aligned}$$

Therefore $H(Q) = J(0) - G(Q) \cdot J(0)$. Substituting for $H(Q)$ in (17) yields

$$\begin{aligned} U(Y, Q) &= G(Q) \cdot J(Y) + J(0) - G(Q) \cdot J(0) \\ &= F(Y) \cdot G(Q) + J(0), \end{aligned}$$

where $F(Y) = J(Y) - J(0)$. Rescaling U by subtracting $J(0)$ yields the multiplicative model, equation (6). **Q.E.D.**

Notes

1. More precisely, the necessary assumptions of generic utility theory are implied by the stronger theories listed in the text. The (nonnecessary) existential assumptions are not implied by these theories, but they are very plausible in the context of these stronger theories. See the remarks on prospect theory for further discussion of this point.
2. Kahneman and Tversky's (1979) statement of prospect theory did not specify what rule governed the case described in equation (5), but equation (5) is implied by the 1979 theory, if one also assumes that (x, p, y) and $(y, 1 - p, x)$ are equally preferred (compare, Miyamoto, 1987). As this assumption is consistent with the spirit of Kahneman and Tversky's analysis, I treat it here as part of prospect theory.
3. This statement may concede too much. Depending on how one axiomatizes prospect theory, the existential axioms of prospect theory would imply Axioms 8 and 9 of generic utility theory. Only Axiom 10 is not ordinarily assumed in prospect theory, although it is quite plausible from that standpoint.
4. The notation used by Pliskin et al. (1980) for the multiplicative model was more complicated than the notation adopted here.
5. There are infinitely many possible survival durations; they are not claimed to be infinitely long or arbitrarily long.
6. Note that the definition of purely irregular subjects motivates the terminological alteration of prospect theory according to which (x, p, y) is classified as irregular if $x = r$ or $y = r$. The key idea is that prospect theory claims that a purely irregular subject evaluates every stimulus gamble by a single rule. If we had not altered the classification of irregular gambles in prospect theory, we would be forced to define "purely irregular" subjects in terms of both regular and irregular gambles, and the present discussion would appear more complicated.

References

Becker, J.L., and R.K. Sarin (1987). "Lottery Dependent Utility." *Management Science*, 33, 1367–1382.

Bell, D.E. (1982). "Regret in Decision Making Under Uncertainty." *Operations Research*, 30, 961–981.

Chew, S., and K. MacCrimmon (1979). "Alpha-nu Choice Theory: A Generalization of Expected Utility Theory." Manuscript faculty of Commerce. University of British Columbia.

Chew, S. (1983). "A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox." *Econometrica*, 51, 1065–1092.

Chew, S. (1984). "An Axiomatization of the Rank Dependent Quasilinear Mean Generalizing the Gini Mean and the Quasilinear Mean." Preprint. Department of Political Economy. The Johns Hopkins University, Baltimore, MD.

Dyer, J.S., and R.K. Sarin (1979). "Measurable Multiattribute Value Functions." *Operations Research*, 27, 810–822.

Edwards, W. (1962). "Subjective Probabilities Inferred from Decisions." *Psychological Review*, 69, 109–135.

Fishburn, P.C. (1965). "Independence in Utility Theory with Whole Product Sets." *Operation Research*, 13, 28–45.

Fishburn, P.C. (1983). "Transitive Measurable Utility." *Journal of Economic Theory*, 31, 293–317.

Kahneman, D., and A. Tversky (1979). "Prospect Theory: An Analysis of Decision Under Risk." *Econometrica*, 47, 276–287.

Kahneman, D., and A. Tversky (1984). "Choices, Values, and Frames." *American Psychologist*, 39, 341–350.

Karmarkar, U.S. (1978). "Subjectively Weighted Utility: A Descriptive Extension of the Expected Utility Model." *Organizational Behavior and Human Performance*, 21, 61–72.

Keeney, R.L., and H. Raiffa (1976). *Decisions with Multiple Objectives*. Wiley, New York.

Krantz, D.H., Luce, R.D., Suppes, P., and A. Tversky (1971). *Foundations of Measurement*, (Vol. 1). Academic Press, New York.

Loomes, G., and L. McKenzie (1989). "The Use of QALYs in Health Care Decision Making." *Social Science & Medicine*, 28, 299–308.

Loomes, G., and R. Sugden (1982). "Regret Theory: An Alternative Theory of Rational Choice Under Uncertainty." *The Economic Journal*, 92, 805–824.

Luce, R.D. (1988). "Rank-dependent, Subjective Expected-utility Representations." *Journal of Risk and Uncertainty*, 1, 305–332.

Luce, R.D. (1992). "Where Does Subjective Expected Utility Fail Descriptively?" *Journal of Risk and Uncertainty*, 5, 5–27.

Luce, R.D. (1990). "Rational Versus Plausible Accounting Equivalences in Preference Judgments." *Psychological science*, 1, 225–234.

Luce, R.D., and P.C. Fishburn (1991). Rank- and Sign-dependent Linear Utility Models for Finite First-order Gambles." *Journal of Risk and Uncertainty*, 4, 29–59.

Luce, R.D., and L. Narens (1985). "Classification of Concatenation Measurement Structures According to Scale Type." *Journal of Mathematical Psychology*, 29, 1–72.

Luce, R.D., and H. Raiffa (1957). *Games and Decisions: Introduction and Critical Survey*. Wiley, New York.

MacCrimmon, K.R., and S. Larsson (1979). "Utility Theory: Axioms or 'Paradoxes.'" In M. Allais & O. Hagen (Eds.), *Expected Utility and the Allais Paradox*. Reidel, Dordrecht, The Netherlands.

Machina, M.J. (1989). "Dynamic Consistency and Non-expected Utility Models of Choice Under Uncertainty." *Journal of Economic Literature*, XXVII, 1622–1668.

McNeil, B.J., and S.G. Pauker (1982). "Optimizing Patient and Societal Decision Making by the Incorporation of Individual Values." In R.L. Kane and R.A. Kane, (Eds.), *Values and Long-term Care* (215–230). D.C. Heath, Lexington, MA.

Mehrez, A., and A. Gafni (1989). "Quality-adjusted Life Years, Utility Theory, and Healthy-years Equivalents." *Medical Decision Making*, 9, 142–149.

Miyamoto, J.M. (1985). *The Utility of Survival Duration and Health Quality: A Conjoint Measurement Analysis* Publication No. 8512473. University Microfilms, Ann Arbor, MI.

Miyamoto, J.M. (1987). "Constraints on the Representation of Gambles in Prospect Theory." *Journal of Mathematical Psychology*, 31, 410–418.

Miyamoto, J.M. (1988). "Generic Utility Theory: Measurement Foundations and Applications in Multiattribute Utility Theory." *Journal of Mathematical Psychology*, 32, 357–404.

Miyamoto, J.M., and S.A. Eraker (1985). "Parameter Estimates for a QALY Utility Model." *Medical Decision Making*, 5, 191–213.

Miyamoto, J.M., and S.A. Eraker (1988). "A Multiplicative Model of the Utility of Survival Duration and Health Quality." *Journal of Experimental Psychology: General*, 117, 3–20.

Miyamoto, J.M., and S.A. Eraker (1989). "Parametric Models of the Utility of Survival Duration: Tests of Axioms in a Generic Utility Framework." *Organizational Behavior and Human Decision Processes*, 44, 166–202.

Narens, L., and R.D. Luce (1986). "Measurement: The Theory of Numerical Assignments." *Psychological Bulletin*, 99, 166–180.

Pliskin, J.S., Shepard, D.S., and M.C. Weinstein (1980). "Utility Functions for Life Years and Health Status." *Operations Research*, 28, 206–224.

Quiggin, J. (1982). "A Theory of Anticipated Utility." *Journal of Economic Behavior and Organization*, 3, 324–343.

Sarin, R.K. (1989). "Analytical Issues in Decision Methodology." I. Horowitz (Ed.), *Decision and Organization Theory*. Kluwer-Nijhoff Academic Publishers, Boston, Mass.

Segal, U. (1984). "Nonlinear Decision Weights with the Independence Axiom." Working Paper 353. Department of Economics. University of California, Los Angeles.

Slovic, P., and S. Lichtenstein (1983). "Preference Reversals: A Broader Perspective." *American Economic Review*, 73, 596–605.

Stevenson, M.K. (1986). "A Discounting Model for Decisions with Delayed Positive or Negative Outcomes." *Journal of Experimental Psychology: General*, 11, 131–154.

Tversky, A., and D. Kahneman (1981). "The Framing of Decisions and the Psychology of Choice." *Science*, 211, 453–458.

Von Neumann, J., and O. Morgenstern (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Von Winterfeldt, D., and W. Edwards (1986). *Decision Analysis and Behavioral Research*. Cambridge University Press, New York.

Wakker, P. (1989a). *Additive Representations of Preferences: A New Foundation of Decision Analysis*. Kluwer, Dordrecht, The Netherlands.

Wakker, P. (1989b). "Continuous Subjective Expected Utility with Non-additive Probabilities." *Journal of Mathematical Economics*, 18, 1–27.

Weber, M., and C. Camerer (1987). "Recent Developments in Modelling Preferences Under Risk." *OR Spektrum*, 9, 129–151.

Weinstein, M.C., Fineberg, H.V., Elstein, A.S., Frazier, H.S., Neuhauser, D., Neutra, R.R., and B.J. McNeil (1980). *Clinical decision analysis* W.B. Saunders, Philadelphia, PA.

Yaari, M.E. (1987). "The Dual Theory of Choice Under Risk." *Econometrica*, 55, 95–115.

The scan of this document produced some black patches on the margins. To avoid printing these black patches (thereby wasting ink), I covered them with blank annotation boxes. If you print this document, print the annotations along with the document, i.e., go to the \File \Print menu of Acrobat, and choose "Document and Markups" in the "Comments and Forms" box. This will prevent the printing of the black patches. (If you don't do this, the black patches will appear in the printed document because the opaque white annotations that cover them won't be printed.)