SUPPLEMENTAL MATERIAL:
Highly Nonlinear Wave Propagation in Elastic Woodpile Periodic Structures

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EXPERIMENTAL SETUP

Figure 5 shows digital images of the experimental woodpile setup constructed in this study. The chain is supported by vertically-standing guiding rods, which constrain the horizontal, in-plane motions of the cylindrical elements and prevent buckling of the chain. To align the rods along the chain in parallel, we position soft polyurethane foam (firmness: 4.1 - 6.2 kPa under 25% deflection) between the tips of the rods. We find that the effect of the supporting foam on the cylinders’ motions during wave propagation is negligible, because of the orders-of-magnitude lower stiffness of the foams compared to the rigidity of the cylinders or the contact stiffness of the vertically stacked cylindrical rods. As a result, we clearly observe the nanopteron tails with the Laser Doppler vibrometer despite the fact that the amplitude of the vibration is extremely small (on the order of tens of nm).

PARAMETER DETERMINATION FOR DEM

For the construction of the DEM, we need to determine the parameters of a unit cell in terms of its discretized mass and spring coefficients (e.g., \(M, m_1, \) and \(k_1\) as shown in the boxed region of Fig 1 (b)). In this study, these parameters are calculated via an optimization process based on the finite element method (FEM) (details are described in the reference [1]). For example, Fig. 6(a) shows the temporal profiles of numerically simulated contact forces in the case of a 40 mm woodpile granular crystal. From the FEM result, it is evident that this locally resonating structure develops a modulated shape of propagating waves, which is in agreement with the experimental results as presented in Fig. 2(b). For the optimization process, we extract the overall shape of the modulated waveform by taking the maximum values of overlapped contact force (see red markers in Fig. 6(a)). The next step involves tuning the parameters of the DEM to match this baseline curve obtained by the FEM. For this, we begin with an

FIG. 5: Experimental setup of a woodpile granular crystal composed of orthogonally stacked 40 mm rods. A magnified view is shown on the right.
FIG. 6: Contact force profiles for a 40 mm woodpile granular crystal from (a) FEM simulation and (b) DEM simulation with optimized parameters.

TABLE I: Discretized mass distribution and spring coefficients in woodpile granular crystals.

<table>
<thead>
<tr>
<th>Rod length</th>
<th>Total mass (g)</th>
<th>$M$ (g)</th>
<th>$m_1$ (g)</th>
<th>$m_2$ (g)</th>
<th>$k_1$ (kN/m)</th>
<th>$k_2$ (kN/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 mm</td>
<td>0.866</td>
<td>0.423</td>
<td>0.443</td>
<td>-</td>
<td>26267</td>
<td>-</td>
</tr>
<tr>
<td>40 mm</td>
<td>1.732</td>
<td>0.838</td>
<td>0.894</td>
<td>-</td>
<td>3961</td>
<td>-</td>
</tr>
<tr>
<td>80 mm</td>
<td>3.464</td>
<td>1.00</td>
<td>1.80</td>
<td>0.664</td>
<td>532</td>
<td>6423</td>
</tr>
</tbody>
</table>

initial mass distribution, $M$ and $m_1$, and compute the stiffness of $k_1$ based on $\omega_0 = \sqrt{\kappa(1 + 1/\nu)}$, where $\kappa = k_1/\beta_c$ and $\nu = m/M$. Note that $\omega_0$ is a resonant frequency of the unit cell (i.e., 40 mm rod) calculated from the FEM, and $\beta_c$ is given simply by the Hertzian contact law. Based on these mass and stiffness values, we simulate the wave propagation via the DEM and obtain the modulated waveforms as shown in Fig. 6(b). We then compare it with the baseline data and iterate this process until we get optimized parameters that yield the best match between the FEM and the DEM. The optimized parameters for discretized masses and spring coefficients are summarized in Table I for woodpile granular chains with rod lengths of 20, 40, and 80 mm. Note, from all the possible internal vibration modes of the rods, only the symmetric bending modes are relevant to the dynamics due to symmetry considerations [1].

FORMATION OF NANOPTERA IN A LONG GRANULAR CHAIN

Figure 7(a) shows a spatio-temporal map of particle velocities in a long chain consisting of 650 particles. After the striker impacts the first particle, we observe the nanopetronic tail is formed immediately within the first 15 particles (see the inset of Fig. 7(a)) given the setup described in the manuscript. This justifies the claim that the length of 40 particle chains (chosen for the numerical and experimental verifications of nanoptera in this study) is sufficient to observe the propagation of the weakly nonlocal solitary waves. Performing the 2D FFT of the nanopetronic tails (corresponding to the boxed region with a dashed line in Fig. 7(a)) demonstrates that the nanopetonic tail has a single frequency and wavenumber associated with it, as shown in Fig. 7(b). The extracted frequency and wavenumber are plotted in Fig. 3 of the main manuscript, and we find that the nanopetron propagates with a constant wave speed identical to the leading solitary wave.

In addition to the formation of nanoptera, we also witness the propagation of secondary solitary waves (see the trailing wave packet in Fig. 7(a)). The formation of multiple solitary wave packets is due to the fact that the striker mass is larger than the mass of the unit particle in the chain. The high inertia of the striker induces multiple collisions with the chain, and this can result in a train of solitary waves [2]. Therein, the speed of the second solitary wave is lower than that of the primary solitary wave due to its small amplitude. This can be verified by the different slopes between the primary and secondary solitary waves in Fig. 7(a). It should be noted that the second solitary wave does not affect the formation and propagation of the nanopetonic tail (i.e., its frequency and wavenumber are not changed by the second solitary wave).
FIG. 7: (a) Spatio-temporal map of particle velocity of a chain consisting of 650 particles (striker velocity: 1.97 m/s) and (b) 2D FFT of the nanopteronic tails in the boxed region in (a).

FIG. 8: (a) Wavenumber and (b) speed of nanopteronic tails (i.e., wings) as a function of various striker velocities (20 mm rods). Blue curves denote DEM results, while the star markers are experimental data measured from four different striker velocities: [0.99 1.40 1.97 2.42] m/s. The wave speed is also calculated alternatively by tracing the solitary waves (i.e., core part of the nanopteron) as marked by hollow circles.

EFFECT OF STRIKER VELOCITY VARIATION

The speed of solitary waves in a discrete chain connected with Hertzian-type contact depends on the wave amplitude, i.e., \( c \propto F_m^{1/6} \) where \( F_m \) is the force magnitude of propagating waves. This force magnitude is strongly dependent on striker velocity \([3]\). Thus, it can be interpreted that the speed of nanopteron (i.e., both of the solitary wave part and tail) increases as we impose a higher striker velocity on the woodpile granular crystal. This is because, as noted above, the wave speed of the nanopteronic tail is the same as the speed of the core, namely \( c = \omega_0/k_0 \), where \( \omega_0 \) and \( k_0 \) are the frequency and wavenumber of the nanopteronic tail, respectively. We numerically verify this using the 40-particle chain composed of 20 mm rods under various striker velocities (Fig. 8). Specifically, we first compute velocity profiles of all particles in the chain using the DEM. Based on the velocity profiles of the nanopteronic tails, we obtain wavenumbers and frequencies by conducting FFT in the space- and time-domain, respectively. The wavenumber varies as a function of striker velocity as shown in Fig. 8(a), while the extracted frequencies coincide with the natural frequency of the cylindrical elements. The wave speed is obtained directly from \( \omega_0/k_0 \) (blue curve in Fig. 8(b)). This is in agreement with the speed of the core part of the nanopteron (i.e., solitary waves’ speed) as marked by hollow circles in Fig. 8(b). Experimental results corroborate the numerical simulations by the DEM (star markers in Fig. 8).
To investigate the effect of dissipation on the nanopteronic tails, we consider two different damping models. One is the damping between the primary and auxiliary masses (Fig. 9(a)) to account for the internal material losses during bending vibrations of the rods. The other is the damping among primary masses (Fig. 9(b)) to consider dissipative losses in contact among the cylindrical rods. A simple dash-pot model is used in the first damping case, while for the second damping model, we use the discrete element model of [4] to implement inter-particle dissipation. We assign arbitrary values for damping parameters, and we focus on the general attenuation trends of the nanopteronic tails in both cases.

Our numerical simulations indicate that the damping between primary and auxiliary masses results in the reduction of the amplitude of both the primary pulses and the nanopteronic tail (Fig. 9(c) and (e)). However, the overall pattern of the oscillatory nanopteronic tail is identical to that of the non-dissipative case described in the main manuscript. If we consider the damping only among the primary masses, we find that the system develops gradually increasing nanopteronic tails, while the amplitude of the primary pulses decreases as the wave propagates (Fig. 9(d) and (f)). This suggests that energy from the primary pulse is being redistributed towards the nanopteronic tail oscillations, a feature that merits further investigation in its own right in future work. Nonetheless, the wave propagation observed in both models incorporating the damping, in principle, clearly demonstrates the persistence of the formation of nanoptera despite the damping.

**EFFECT OF DAMPING ON NANOPTERA**

In the absence of the local resonators, Eq. (1) possesses exact traveling wave solutions [5–7]. Two classical approaches to obtain an analytical approximation to the evolution of the traveling wave in the presence of perturbations are the map approach [8] and the binary collision approximation (BCA) [9]. In this section, we amend these two methods in order to study the traveling waves in the presence of the local resonators. It should be noted here that the map approach presented below in the presence of resonators was already developed in the work of [10], whose discussion we adapt for our present comparison with the experimental results.

**SEMI-ANALYTICAL DESCRIPTION OF WAVE PROPAGATION**

In the absence of the local resonators, Eq. (1) possesses exact traveling wave solutions [5–7]. Two classical approaches to obtain an analytical approximation to the evolution of the traveling wave in the presence of perturbations are the map approach [8] and the binary collision approximation (BCA) [9]. In this section, we amend these two methods in order to study the traveling waves in the presence of the local resonators. It should be noted here that the map approach presented below in the presence of resonators was already developed in the work of [10], whose discussion we adapt for our present comparison with the experimental results.
A Map Approach

Ignoring the boundary effects (i.e., considering an infinite chain), we write the equations of motion in terms of the strain variables $r_i = u_{i-1} - u_i$ and $s_i = v_{i-1} - v_i$

\[
\ddot{r}_i = [r_{i-1}]^{3/2}_+ - 2[r_i]^{3/2}_+ - [r_{i+1}]^{3/2}_+ + \kappa (s_i - r_i),
\]

\[
\ddot{s}_i + \omega^2 s_i = \omega^2 r_i,
\]

where $\omega^2 = \omega_0^2 - \kappa = \kappa/\nu$. Viewing the right hand side of Eq. (9) as an inhomogeneity suggests expressing the solution in terms of the Green's function of the operator $[\frac{d^2}{d\tau^2} + \omega^2]

\]

Substituting this expression into Eq. (8) yields

\[
\ddot{r}_i = [r_{i-1}]^{3/2}_+ - 2[r_i]^{3/2}_+ + \frac{\kappa}{2} \left( \omega \int_{-\infty}^t \sin(\omega(t-\tau)) r_i(\tau) d\tau \right) - \kappa r_i.
\]

In the absence of the local resonators ($\kappa = 0$), it is well known that Eq. (10) possesses solitary wave solutions [5–7]. One of the earlier results of Nesterenko relates the amplitude $A$ of the traveling wave to its velocity [11]

\[
r_i(t) = A \cdot S(A^{1/4} t - i)
\]

where $S(\xi)$ is the profile of the wave. While there are several analytical approximations of this profile, based on e.g. a long wavelength approximation [11–13], we use the one recently developed in [13], which has the form

\[
S(\xi) = \left( \frac{1}{q_0 + q_2 \xi^2 + q_4 \xi^4 + q_6 \xi^6 + q_8 \xi^8} \right)^2
\]

where $q_0 \approx 0.8357, q_2 \approx 0.3669, q_4 \approx 0.0831, q_6 \approx 0.0125$ and $q_8 \approx 0.0011$. If we assume that traveling wave form is maintained in the presence of the local resonators, but has a possibility decaying amplitude, then we have

\[
r_i(t) = A_i S(A_i^{1/4} t - i),
\]

where $S$ is given by Eq. (11). Substituting this expression into Eq. (10) yields

\[
A_i^{3/2} S_i'' = [A_{i-1} S_{i-1}]^{3/2}_+ - 2[A_i S_i]^{3/2}_+ + [A_{i+1} S_{i+1}]^{3/2}_+ + \kappa A_i \left( \omega \int_{-\infty}^t \sin(\omega(t-\tau)) S(A_i^{1/4} t - i) d\tau \right) - \kappa A_i S_i
\]

where we used the notation $S_i = S(A_i^{1/4} t - i)$. Since the peak of the traveling wave occurs for $T_i = iA_i^{-1/4}$, and the tail of the solutions asymptotes monotonically to zero, we can write Eq. (13) as a discrete map by integrating from $(-\infty, T_i)$

\[
0 = A_i^{3/2} \int_{-\infty}^{T_i} S^{3/2}(A_i^{1/4} t + i - 1) dt - 2A_i^{3/2} \int_{-\infty}^{T_i} S^{3/2}(A_i^{1/4} t + i) dt + A_{i+1}^{3/2} \int_{-\infty}^{T_i} S^{3/2}(A_{i+1}^{1/4} t + i + 1) dt
\]

\[
+ \kappa A_i \left( \omega \int_{-\infty}^{T_i} \int_{-\infty}^t \sin(\omega(t-\tau)) S(A_i^{1/4} t - i) d\tau dt \right) - \kappa A_i \int_{-\infty}^{T_i} S(A_i^{1/4} t + i) dt
\]

where we assumed $A_i > 0$ and $S_i > 0$. Several of the integrals appearing in this expression can be evaluated after a change of variable $t \rightarrow A_i^{1/4} t + i$. Thus we define

\[
f_1 = \int_{-\infty}^{0} S^{3/2}(t-1) dt \approx 2.5051,
\]

\[
f_2 = \int_{-\infty}^{0} S^{3/2}(t) dt \approx 1.3215,
\]

\[
f_3 = \int_{-\infty}^{0} S^{3/2}(t+1) dt \approx 0.1379,
\]

\[
g = \int_{-\infty}^{0} S(t) dt \approx 1.2551
\]
FIG. 10: Arrival times and corresponding strains $r_i$ of the initial traveling wave front in the 40 mm rod set up as predicted by the DEM simulation (black line), the discrete map (blue points) and the TCA (red circles). The squares with error bars are the experimentally measured values, where the standard deviations were computed using five experimental runs.

such that

$$0 = A_i^{5/4}f_1 - 2A_i^{5/4}f_2 + A_i^{5/4}f_3 + \kappa A_i \left( \omega \int_{-\infty}^{T_i} \int_{-\infty}^{t} \sin(\omega(t-\tau))S(A_i^{1/4} \tau - i) d\tau dt \right) - \kappa A_i^{3/4}g.$$

Assuming that local resonators create small perturbations of size $\epsilon$ such that $A_i - 1 = A_i + O(\epsilon)$ finally yields a one-dimensional discrete map

$$A_{i+1} = \left( \frac{A_i^{5/4}f_1 + \kappa A_i \left( \omega \int_{-\infty}^{T_i} \int_{-\infty}^{t} \sin(\omega(t-\tau))S(A_i^{1/4} \tau - i) d\tau dt \right) - \kappa A_i^{3/4}g}{2f_2 - f_3} \right)^{4/5}.$$

(14)

By construction, the arrival time at the $i$th lattice site will be $T_i = iA_i^{-1/4} + \phi_0$, where $\phi_0$ is used to calibrate the position of the first peak. To check the validity of this approximation, we compare the approximate decay of the peaks and corresponding arrival times to the DEM simulations. In Fig. 10 we consider the parameter values determined by the optimization procedure described in Table I for a rod length of 40 mm. In this case, the arrival time and strain value $r$ of the initial traveling wave front compare favorably among the experiment, full DEM calculation, and map based on Eq. (14).

In the main text, three types of behavior are observed based on rod length, namely (i) the spontaneous formation (and steady propagation) of the nanopteron (found in the 20 mm rod), (ii) the potential breathing of the traveling solitary waves (in the 40 mm rod) or (iii) the decay of the solitary waves (in the 80 mm rod) without relying on internal damping. This is due to the fact that each rod length yields different values of the resonant frequency. Thus, varying the resonant frequency will allow one to observe behavior ranging from modulated waves (i.e., breathing) to nanoptera. For example, Fig. 11 compares simulations of the DEM model and the map (Eq. (14)) for other values of the local resonant frequency ($\omega_0$) by changing nondimensional stiffness $\kappa$, while keeping the other parameter values fixed. The modulation effect, which is due to the presence of secondary waves, is not captured by the map approach, which only takes into account the effects of the initial wave. Smaller values of the parameter $\kappa$ correspond to slower decay of the initial traveling wave front, which is captured well by the map approach. Likewise, as $\kappa$ is increased, the decay rate increases. However, in the full DEM model, the secondary wave effects have an immediate impact, which the map approach fails to predict, see Fig 11(c) and (d). In panel (d), one sees the development of the nanopteron. Note that the role of the secondary wave in numerical observations was also discussed in [10]. In Figs. 10 and 11 the predictions based on a ternary collision approximation (TCA) are also shown, which is detailed in the following section.
An Extension of the BCA: A Ternary Collision Approximation

The main purpose of the binary collision approximation (BCA) is to offer an approximation of the solitary wave velocity, or rather, the arrival time of the pulse for a given bead. The idea is to solve a simplified set of equations over a time scale where the dominant effects are between two adjacent beads only. In some cases, the resulting simplified equations can be solved exactly, the solution of which can be used to initialize the following step. Stringing the approximations together yields an approximation over the desired time span.

Employing the same procedure for the DEM model, Eqs. (1) and (2) in the main manuscript leads to a ternary collision approximation (since two primary masses and a local resonator are now involved). The simplified equations in the rescaled variables have the form

\[
\ddot{u}_0 = \left[ u_0 - u_1 \right]^{3/2} - \kappa (u_0 - v),
\]
\[
\ddot{u}_1 = \left[ u_0 - u_1 \right]^{3/2},
\]
\[
\ddot{v} = \omega^2 (u_0 - v).
\]

where \( u_0 \) is the displacement of the impacted bead, \( u_1 \) is the displacement of the bead adjacent to it, and \( v \) is the displacement of the local resonator of the impacted bead. Now define \( r = u_0 - u_1 \) and \( s = u_0 - v \). We have then

\[
\ddot{r} = -2[r]^{3/2} + \kappa s,
\]
\[
\ddot{s} = -[r]^{3/2} - (\kappa + \omega^2) s.
\]
The goal is to use these equations to obtain an approximation for the arrival time of the pulse. Recall that the arrival time at the $i$th site corresponds to the time where $r_i(t)$ obtains its first local maximum, or equivalently when the velocities of the adjacent beads become equal, since by definition $r_i = u_{i-1} - u_i$. For the first iteration, we use the initial value $r(0) = s(0) = 0$ and $\dot{r}(0) = \dot{s}(0) = V_0$, where $V_0$ is the velocity of the striker bead. Upon solving system (15), we define the first arrival time $t_1$ such that $\dot{r}(t_1) = 0$. For the subsequent iterations, we use the initial value $r(t) = s(t) = 0$ and $\dot{r}(t) = \dot{s}(t) = V_i$. In the absence of the local resonator, the traveling wave does not decay, and hence $V_i = V_0$. We assume however, that this resonator will indeed cause the peak to decay. We estimate the decay based on the first iteration. Namely, we define

\[ V_i = V_0 a^{i-1}, \quad a = \frac{\dot{r}(t_1; \kappa)}{\dot{r}(t_1; 0)} \]

where $r(t_1; 0)$ is the solution of (15) at time $t = t_1$ with the resonator parameter $\kappa = 0$ and $r(t_1; \kappa)$ is the solution of (15) at time $t = t_1$ with the resonator parameter $\kappa \neq 0$. In other words, $a$ is the decay between the first and second peak, and we assume the decay rate remains constant. Since the TCA is an iterative procedure, the $i$th arrival time $t_i$ is based on an initial time of $t = 0$. Thus, the actual arrival time at site $i$ is $T_i = \sum_{n=0}^{i-1} t_n$. The pulse arrival times based on the TCA using the parameter values corresponding to the 40 mm rod are shown as the red circles in Fig. 10, where we found $a \approx 0.8$. This method also compares favorably to the experiment, full DEM simulation and the map approach. To estimate the amplitude of peak $i$, we used the formula $A_0 a^i$, where $A_0$ is the amplitude of the first peak computed from the DEM. These predictions are shown as red circles in Fig. 11.

Here, the TCA reduced an infinite dimensional system of ODEs into a system of two second order equations (see e.g. Eq. (15)). However, the simplified system does not lend itself to a closed form analytical solution, and thus we resorted to numerical simulations of Eq. (15). Nonetheless, the favorable agreement between the TCA and full DEM simulations demonstrates that the predominant features of the initial traveling wave front at a given time are adequately captured by the simplified model bearing two primary masses and a local resonator.